Binary Search and Worst-Case Analysis

Yufei Tao

Department of Computer Science and Engineering
Chinese University of Hong Kong

A significant part of computer science is devoted to understanding the power of the RAM model in solving specific problems. Every time we discuss a problem in this course, we will learn something new.

Today's lecture is about the dictionary search problem. We will learn not only a fast algorithm for solving this problem, but also a method called worst-case analysis for measuring the quality of an algorithm.

The Dictionary Search Problem

Problem Input:

In the memory, a set S of n integers have been arranged in ascending order at the memory cells from address 1 to n. The value of n has been placed in Register 1 of the CPU. Another integer v has been placed in Register 2 of the CPU.

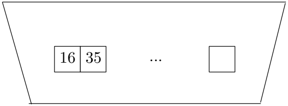
Goal:

Design an algorithm to determine whether v exists in S.

> Note that we have not specified how your algorithm should indicate the outcome. This is up to you. For example, you may store 0 in a certain register to signify "no", and 1 for "yes".

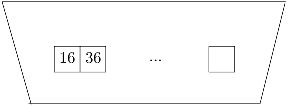
We will refer to the value of n as the **problem size**.

A “yes”-input with $n = 16$



5	9	12	17	26	28	35	38	41	47	52	68	69	72	83	88														

A “no”-input with $n = 16$



5	9	12	17	26	28	35	38	41	47	52	68	69	72	83	88														

The First Algorithm

Let n be Register 1, and v be Register 2.

Simply read the memory cell of address i, for each $i \in [1, n]$ in turn. If any of those cells equals v, return yes. Otherwise, return no.

The above is a concise, but clear, description of the same algorithm as in the pseudocode of the next slide.

The First Algorithm in Pseudocode

```
Let n be register 1, and v be register 2
register i ← 1, register one ← 1
repeat
    read into register x the memory cell at address i
    if x = v then
        return "yes" (by writing 1 to a register)
    i ← i + one (effectively increasing i by 1)
until i > n
return "no" (by writing 0 to a register)
```

Running Time of the First Algorithm

How much time does the algorithm require? The answer depends on the problem input. Here are two extreme cases:

- If v is the first element in S (i.e., the integer in the memory cell of address 1), the algorithm has running time 5.
- If we are given a "no"-input, then the algorithm has running time $4n + 3$.

In computer science, it is an art to design algorithms with performance guarantees. In our scenario, this amounts to the question: what is the largest running time on the worst input with n integers?

This gives rise to an important notion in the next slide.

Worst-Case Running Time

The **worst-case cost** (or **worst-case time**) of an algorithm under a problem size n, is defined to be the largest running time of the algorithm on all the (possibly an infinite number of) inputs of the same size n.

Example

Our algorithm has worst-case time $f_1(n) = 4n + 3$.

In other words, no matter how you design the input set S of n integers, the algorithm always terminates with a cost at most $4n + 3$. This is its performance guarantee on every n.

Next, we will see another algorithm with much better worst-case time, namely, the binary search algorithm.

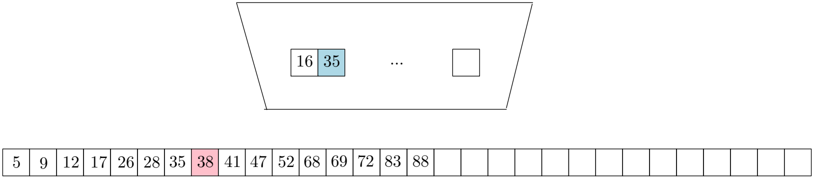
Binary Search

We will utilize the fact that S has been stored in ascending order. Let us compare v to the element x in the middle of S (i.e., the $(n/2)$-th).

- If $v = x$, we have found v, and thus, can terminate.
- If $v < x$, we can immediately forget about the second half of S.
- If $v > x$, forget about the first half.

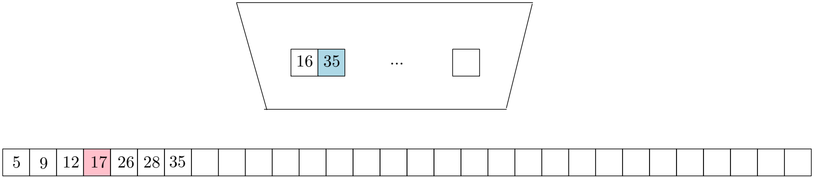
In the 2nd and 3rd cases, we have at most $n/2$ elements left.
Then, repeat the trick on those elements!

Binary Search



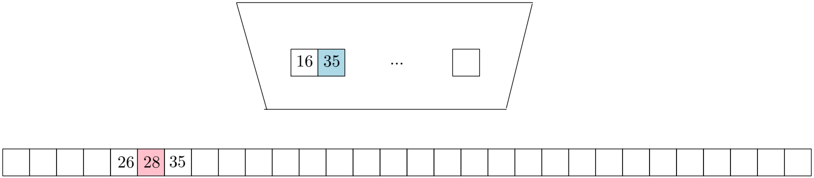
Conceptually discard the second half of S.

Binary Search



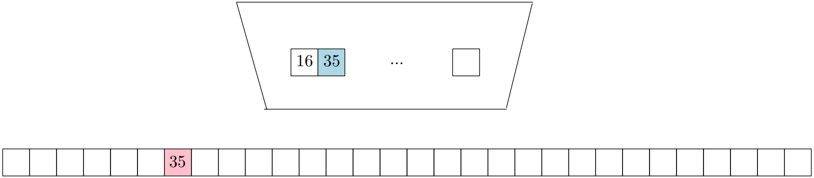
Conceptually discard the first half of what is shown.

Binary Search



Conceptually discard the first half of what is shown.

Binary Search



Found.

Binary Search in Pseudocode

```
let n be register 1, and v be register 2
register left ← 1, right ← n
repeat
    register mid ← (left + right)/2
    if the memory cell at address mid = v then
        return "yes"
    else if the memory cell at address mid > v then
        right = mid − 1
    else
        left = mid + 1
until left > right
return "no"
```

Worst-Case Time of Binary Search

Let us call the integers whose memory addresses are from *left* to *right* as active elements.

Refer to Lines 3-10 as an iteration. Each iteration performs at most 7 atomic operations (try verifying this yourself).

Worst-Case Time of Binary Search

How many iterations are there? After the first iteration, the number of active elements is at most $n/2$. After another, the number is at most $n/4$. In general, after i iterations, the number drops to at most $n/2^i$.

Suppose that there are h iterations in total. It holds that h is the smallest integer satisfying (think: why?)

$$\frac{n}{2^h} < 1$$

which gives $h = \lceil \log_2(n+1) \rceil$.

Worst-Case Time of Binary Search (cont.)

In each iteration we perform only a constant number of operations — we will not analyze this constant precisely, except for pointing out a loose upper bound of 10.

The worst-case time of binary search is at most $f_2(n) = 10(1 + \log_2 n)$.

When n is large, this running time is much lower than the time $4n + 3$ of our first algorithm.

In this lecture, we have got a taste of what computer science is like. We are seldom satisfied with just finding an algorithm that can correctly solve a problem. Instead, our goal is to design an algorithm with a strong performance guarantee, i.e., you must prove that it runs fast even in the worst case.