CSCI2100: Regular Exercise Set 8

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Problem 1. Prove: A tree with n nodes has $n-1$ edges.

Solution. We will prove the claim by induction. The case of $n = 1$ is obviously true. Now suppose that the claim is true for all $n \leq k$, we now proceed to prove that it is also true for $n = k+1$. Let T be a tree with $k+1$ nodes. Remove an arbitrary edge of T. The fact that the original T has no cycle implies that now T has been divided into two trees T_1 and T_2. If T_1 has x nodes, then T_2 has $k+1-x$ nodes. Our inductive claim shows that T_1 has $x-1$ edges and T_2 has $k+1-x-1 = k-x$ edges. Therefore, T has $(x-1)+(k-x)+1 = k$ edges. This completes the proof.

Problem 2 (Max Heap). The binary heap we discussed in the class is called the *min-heap* because of the `delete-min` operation. Conversely, a *max-heap* on a set S of integers aims to support insertions and the following `delete-max` operation:

- `Delete-max`: Reports the largest integer in S, and removes it from S.

Describe how a min-heap can be used to implement a max-heap *without* changing its structure and algorithms. Your max-heap must still use $O(|S|)$ space, and support an insertion and a delete-max operation in $O(\log |S|)$ time.

Solution. To perform an insertion of e, simply insert $-e$ to a min-heap. To perform a delete-max, simply perform a delete-min from the min-heap, and then return the fetched value after negating it.

Problem 3* (Priority Queue with Attrition). Let S be a dynamic set of integers. At the beginning S is empty. We want to support the following operations:

- `Insert-with-Attrition`(e): First removes all integers in S that are greater than e, and then adds e to S.
- `Delete-Min`: Removes and returns the smallest integer of S.

For example, suppose we perform the following sequence of operations:

1. `Insert-with-Attrition`(83)
2. `Insert-with-Attrition`(5)
3. `Insert-with-Attrition`(10)
4. `Insert-with-Attrition`(15)
5. `Insert-with-Attrition`(12)
6. `Delete-Min`
7. `Delete-Min`

After Operation 3, $S = \{5, 10\}$ (note that 83 has been deleted by Operation 2). After Operation 5, $S = \{5, 10, 12\}$. After Operation 6, $S = \{10, 12\}$.

Describe a data structure with the following guarantees:

- At all times, the space consumption is $O(|S|)$.

- Any sequence of n operations (each being an `insert-with-attrition` or `delete-min`) is processed with $O(n)$ time, i.e., $O(1)$ amortized time per operation.

Solution. We simply maintain all the elements of S in a queue Q, where they are arranged in the same order by which they enter S. Given an `Insert-with-Attrition`(e) operation, we keep walking back from the tail of Q until either seeing the first element smaller than e or having exhausted the entire Q. Delete all the elements that (i) are already seen, and (ii) are larger than e. It is important to observe that at this moment all the remaining elements in Q are sorted in ascending order.

To perform a `delete-min`, simply remove the first element of Q.

The cost of `delete-min` is clearly $O(1)$. The cost of `Insert-with-Attrition` equals $O(1+x)$ where x is the number of elements removed. The total cost of all the `Insert-with-Attrition` operations is $O(n)$ because every element can contribute to the x-term only once.

Problem 4 (Textbook Exercise 6.5-9). Suppose that we have k arrays $A_1, A_2, ..., A_k$ of integers, such that each array has been sorted in ascending order. Let n be the total number of integers in those arrays. Describe an algorithm to produce an array that sorts all the n integers in ascending order (you may assume that no integer exists in two arrays). Your algorithm must finish in $O(n \log k)$ time.

For example, suppose that $k = 3$, and that the three arrays are $(2, 23, 32, 35, 37)$, $(5, 10)$, and $(33, 58, 82)$. Then you should produce an array containing $(2, 5, 10, 23, 32, 33, 35, 37, 58, 82)$.

Solution. Insert the smallest element of each array into a binary heap H. This takes $O(k \log k)$ time. Then, repeat the following until H is empty:

- Perform a delete-min. Let e be the element fetched.
- Append e to the output array.
- If e comes from A_i (for some i), obtain the next element from A_i, and insert it into H. If A_i has been exhausted, then do nothing.

Each delete-min and insertion require $O(\log k)$ time because H has at most k elements. There are n delete-min and n insertions. So the total cost is $O(n \log k)$.