ENGG1410-F Tutorial 2

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Problem 1. Gauss Elimination

Consider the following liner system:

$$\begin{aligned} 2y + z &= -8 \\ x - 2y - 3z &= 0 \\ -x + y + 2z &= 3 \end{aligned}$$

Solve it with Gauss Elimination.

Solution

We first obtain the augmented matrix:

$$\begin{bmatrix} 0 & 2 & 1 & -8 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 2 & 3 \end{bmatrix}$$

Solution–cont.

Next, we convert the matrix into row echelon form:

$$\begin{bmatrix} 0 & 2 & 1 & -8 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 2 & 3 \\ 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 1 & 2 & 3 \\ 0 & -1 & -1 & 3 \\ 0 & 2 & 1 & -8 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 1 & 2 & 3 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

Solution–cont.

Now apply back substitution to obtain the solution of x, y, z. Specifically,

$$-z = -2 \quad \Rightarrow \quad z = 2$$

$$-y - z = 3 \quad \Rightarrow \quad y = -5$$

$$-x + y + 2z = 3 \quad \Rightarrow \quad x = -4$$

Therefore, the solution of the linear system is $x = -4, y = -5, z = 2$.

Problem 2. Rank calculation

Calculate the rank of the following matrix:

$$\begin{bmatrix} 0 & 16 & 8 & 4 \\ 2 & 4 & 8 & 16 \\ 16 & 8 & 4 & 2 \\ 4 & 8 & 16 & 2 \end{bmatrix}$$

Problem 3. An Import Property of Ranks

Consider the following 3×5 matrix:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 & 5 & 7 \\ 1 & \sqrt{2} & \sqrt{3} & \sqrt{5} & \sqrt{7} \\ 1 & 2^{1/3} & 3^{1/3} & 5^{1/3} & 7^{1/3} \end{bmatrix}$$

Prove: there must be a column vector that is a linear combination of the other column vectors.

Solution

Proof. Denote by $\boldsymbol{c}_i$ $(i = 1, 2, \cdots, 5)$ the i-th column vector of $\boldsymbol{A}$, since $\boldsymbol{A}$ is a 3×5 matrix, we know that

$$\text{rank}\boldsymbol{A} = \text{rank}\boldsymbol{A}^T \leq 3$$

which implies that the column vectors of $\boldsymbol{A}$ are linearly dependent. In other words, there exist real values $\alpha_1, \cdots, \alpha_5$ such that

- they are not all zero;
- they satisfy $\sum_{i=1}^{5} \alpha_i \boldsymbol{c}_i = \mathbf{0}$.

Suppose $\alpha_k \neq 0$ for some k, then we have:

$$\boldsymbol{c}_k = -\sum_{i=1, i\neq k}^{5} \frac{\alpha_i}{\alpha_k} \boldsymbol{c}_i$$

That said, $\boldsymbol{c}_k$ is a linear combination of the other column vectors. □

In fact, the above conclusion can be generalized, i.e.:

for an $m \times n$ matrix $\boldsymbol{A}$, if $m < n$, then there must be a column vector of $\boldsymbol{A}$ that is a linear combination of the other column vectors.

The proof is similar and left to you as an exercise.

Problem 4. Rank calculation

Consider a plane $z = 2x + 3y$ in 3-dimensional space, suppose there are m points on this plane, and point i has the coordinates (x_i, y_i, z_i), where $i = 1, \cdots, m$. Let

$$\boldsymbol{A} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \vdots & \vdots & \vdots \\ x_m & y_m & z_m \end{bmatrix}$$

Prove: $\text{rank}\boldsymbol{A} \leq 2$.

Solution

Proof. Perfom *elementary column operations* on $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \vdots & \vdots & \vdots \\ x_m & y_m & z_m \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 2x_1 + 3y_1 \\ x_2 & y_2 & 2x_2 + 3y_2 \\ \vdots & \vdots & \vdots \\ x_m & y_m & 2x_m + 3y_m \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 & y_1 & 3y_1 \\ x_2 & y_2 & 3y_2 \\ \vdots & \vdots & \vdots \\ x_m & y_m & 3y_m \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ \vdots & \vdots & \vdots \\ x_m & y_m & 0 \end{bmatrix}$$

Hence, $\text{rank}\boldsymbol{A} = \text{rank}\boldsymbol{A}^T \leq 2$. □

Problem 5. Determinant calculation

Calculate the determinant of the following matrix:

$$\begin{bmatrix} 0 & 4 & -6 \\ 4 & 0 & 10 \\ -6 & 10 & 0 \end{bmatrix}$$

Problem 6. Rank Properties

Prove: $\text{rank}(\boldsymbol{AB}) \leq \text{rank}\boldsymbol{A}$.

Solution

Recall:

- Elementary row operations on a matrix do not change its rank.
- Perform an elementary row operation on a matrix $\boldsymbol{A}$ is equivalent to left-multiplying $\boldsymbol{A}$ by a *row elementary matrix*.
- The rank of a matrix of row echelon form is the number of its non-zero rows.

Solution–cont.

Proof. Denote by $\boldsymbol{A}'$ the row echelon form of $\boldsymbol{A}$, $\boldsymbol{E}_i$ a row elementary matrix, and suppose $\boldsymbol{A}'$ is obtained from $\boldsymbol{A}$ by performing z elementary row operations, i.e.,

$$\boldsymbol{A}' = (\Pi_{i=1}^{z} \boldsymbol{E}_i)\boldsymbol{A} = \boldsymbol{E}\boldsymbol{A}$$

Let rank$\boldsymbol{A}$ = rank$\boldsymbol{A}' = r$, i.e., the first r rows of $\boldsymbol{A}'$ are non-zero, whereas the remaining rows are all zero vectors.

Suppose $\boldsymbol{A'}$ is an $m \times n$ matrix, $\boldsymbol{B}$ is an $n \times p$ matrix, denote the row vectors of $\boldsymbol{A}'$ as $\boldsymbol{r}_1, \cdots, \boldsymbol{r}_m$ in top-down order and the column vectors of $\boldsymbol{B}$ as $\boldsymbol{c}_1, \cdots, \boldsymbol{c}_p$ in left-to-right order.

Solution–cont.

Then, we have

$$A'B = \begin{bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \vdots \\ \boldsymbol{r}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \cdots & \boldsymbol{c}_p \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}_1 \\ \vdots \\ \boldsymbol{r}_r \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \cdots & \boldsymbol{c}_p \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{r}_1 \cdot \boldsymbol{c}_1 & \boldsymbol{r}_1 \cdot \boldsymbol{c}_2 & \cdots & \boldsymbol{r}_1 \cdot \boldsymbol{c}_p \\ \vdots & \vdots & \cdots & \vdots \\ \boldsymbol{r}_r \cdot \boldsymbol{c}_1 & \boldsymbol{r}_r \cdot \boldsymbol{c}_2 & \cdots & \boldsymbol{r}_r \cdot \boldsymbol{c}_p \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

Solution–cont.

Therefore,

$$\begin{aligned}\text{rank}(\boldsymbol{AB}) &= \text{rank}(\boldsymbol{E(AB)}) \\ &= \text{rank}(\boldsymbol{(EA)B}) \\ &= \text{rank}(\boldsymbol{A'B}) \\ &\le r = \text{rank}\boldsymbol{A}\end{aligned}$$

where the first inequality used the fact that performing the elementary row operations indicated by $\boldsymbol{E}$ do not change the rank of $\boldsymbol{AB}$. □

Problem 7. Rank Properties

Let $\boldsymbol{A}$ be a $m \times n$ matrix, $\boldsymbol{B}$ be a $p \times q$ matrix obtained by extracting p rows and q columns of $\boldsymbol{A}$, i.e., $\boldsymbol{B}$ is a submatrix of $\boldsymbol{A}$.
Prove: $\text{rank}(\boldsymbol{B}) \leq \text{rank}(\boldsymbol{A})$.

Solution

Proof. Denote by $\boldsymbol{r}_i$ the i-th row vector of $\boldsymbol{A}$, and $\boldsymbol{r}'_j$ the j-th row vector of $\boldsymbol{B}$, where $i = 1, \cdots, m$ and $j = 1, \cdots, p$. Assume rank$\boldsymbol{B} = r$, then there must be r row vectors of $\boldsymbol{B}$ that are linearly independent, let them be $\boldsymbol{r}'_{x_1}, \boldsymbol{r}'_{x_2}, \cdots, \boldsymbol{r}'_{x_r}$, and the corresponding row vectors of $\boldsymbol{A}$ are $\boldsymbol{r}_{y_1}, \boldsymbol{r}_{y_2}, \cdots, \boldsymbol{r}_{y_r}$, where $x_k \in [1, p]$, $y_k \in [1, m]$, $k \in [1, r]$ and x_k, y_k, k are all integers. Note that $\boldsymbol{r}_{y_k}$ is an expansion of $\boldsymbol{r}'_{x_k}$ for each k.

Then we have

$$\sum_{k=1}^{r} \alpha'_k \boldsymbol{r}'_{x_k} = \mathbf{0} \text{ iff } \alpha'_1 = \alpha'_2 = \cdots = \alpha'_k = 0. \tag{1}$$

Solution–cont.

Hence we must have

$$\sum_{k=1}^{r} \alpha_k \boldsymbol{r}_{y_k} = \mathbf{0} \text{ iff } \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0. \quad (2)$$

Otherwise, set $\alpha'_k = \alpha_k$ for each k will violate (1).

(2) implies rank$\boldsymbol{A} \geq r =$ rank$\boldsymbol{B}$. □