Question 1

Consider the following vectors, where a is a real number:

$$\begin{bmatrix}1 & 8 & 6 & 4 & 0\end{bmatrix}$$
$$\begin{bmatrix}8 & 5 & 7 & 6 & 9\end{bmatrix}$$
$$\begin{bmatrix}4 & 6 & 8 & 7 & 1\end{bmatrix}$$
$$\begin{bmatrix}5 & a & 5 & 3 & a+1\end{bmatrix}$$

a. **(8 marks)** If these vectors are linearly dependent, what should be the value of a?

b. **(8 marks)** Now assume that a takes a value such that these vectors are linearly independent. Consider the following system of homogeneous equations

$$\begin{array}{llllll} x & +8y & +6z & +4u & & = 0 \\ 8x & +5y & +7z & +6u & +9w & = 0 \\ 4x & +6y & +8z & +7u & +w & = 0 \\ 5x & +ay & +5z & +3u & +(a+1)w & = 0 \end{array}$$

Is it possible that a nontrivial solution to the system exists, in which $w = 0$. What should be the value of a if such a solution exists?

SOLUTION

a. $$\begin{bmatrix}1 & 8 & 6 & 4 & 0 \\ 8 & 5 & 7 & 6 & 9 \\ 4 & 6 & 8 & 7 & 1 \\ 5 & a & 5 & 3 & a+1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 8 & 6 & 4 & 0 \\ 0 & -59 & -41 & -26 & 9 \\ 0 & -26 & -16 & -9 & 1 \\ 0 & a-40 & -25 & -17 & a+1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 8 & 6 & 4 & 0 \\ 0 & -59 & -41 & -26 & 9 \\ 0 & 0 & \frac{122}{59} & \frac{145}{59} & -\frac{175}{59} \\ 0 & 0 & 0 & \frac{47a-329}{122} & \frac{19a-133}{122}\end{bmatrix}$$

Since the last row cannot be all zero, these vectors are all linearly independent for all values of a.

b. Now assume $w = 0$. We have $$\begin{bmatrix}1 & 8 & 6 & 4 \\ 0 & -59 & -41 & -26 \\ 0 & 0 & \frac{122}{59} & \frac{145}{59} \\ 0 & 0 & 0 & \frac{47a-329}{122}\end{bmatrix}\begin{bmatrix}x \\ y \\ z \\ u\end{bmatrix} = \begin{bmatrix}0 \\ 0 \\ 0 \\ 0\end{bmatrix}.$$

Therefore, nontrivial solution exists if and only if $$\begin{vmatrix}1 & 8 & 6 & 4 \\ 0 & -59 & -41 & -26 \\ 0 & 0 & \frac{122}{59} & \frac{145}{59} \\ 0 & 0 & 0 & \frac{47a-329}{122}\end{vmatrix} = 0, \text{ or } a = \frac{329}{47} = 7.$$

Question 2

(a) Given the following two matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

Compute the determinants for the following: $det(\mathbf{A})$, $det(\mathbf{B})$, $det(\mathbf{A}^\top \mathbf{B})$.

(b) Balance the following chemical reactions by setting up and solving a homogeneous linear system with respect to unknown variables x_1, x_2, x_3, x_4 that keeps the number of atoms of all kinds identical before and after the reaction. Please also describe the system's coefficient matrix's row and column vector space dimensionality and the system's solution space dimensionality. Note that C is carbon, H is hydrogen, O is oxygen and Cl is chlorine.

$$x_1 \mathrm{CH_3CH_2OH} \quad + \quad x_2 \mathrm{Cl_2} \quad \rightarrow \quad x_3 \mathrm{CCl_3CHO} \quad + \quad x_4 \mathrm{HCl}$$

Answer:

(a) 6, 18, 108 [6 points]

(c) $\mathbf{x}^\top = k[\ 1\ 4\ 1\ 5\ ]\ ^\top$, 3, 3, 1 [10 points]

Question 3. Let

$$\boldsymbol{A} = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$$

(a) (6 marks) Calculate $\boldsymbol{A}\boldsymbol{A}^T$.

(b) (6 marks) Using the result in part (a), or otherwise, find $\boldsymbol{A}^{-1}$

(c) (4 marks) Using the result in part (b), or otherwise, solve the following linear system:

$$\begin{array}{rcrcrcl} 2x_1 & - & 2x_2 & + & x_3 & = & 9 \\ x_1 & + & 2x_2 & + & 2x_3 & = & 0 \\ 2x_1 & + & x_2 & - & 2x_3 & = & 0 \end{array}$$

Solution.

(a)

$$\boldsymbol{A}\boldsymbol{A}^T = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

(b)

To find $\boldsymbol{A}^{-1}$

– Method 1: From part (a), we have

$$\boldsymbol{A}\boldsymbol{A}^T = 9\boldsymbol{I}$$

Thus

$$\boldsymbol{A}^{-1} = \frac{1}{9}\boldsymbol{A}^T = \begin{bmatrix} 2/9 & 1/9 & 2/9 \\ -2/9 & 2/9 & 1/9 \\ 1/9 & 2/9 & -2/9 \end{bmatrix}$$

– Method 2 (Gauss-Jordan Method):

$$\left[\ \boldsymbol{A} \mid \boldsymbol{I}\ \right] \longrightarrow \left[\ \boldsymbol{I} \mid \boldsymbol{A}^{-1}\ \right]$$

– Method 3 (Adjoint Matrix Method):

$$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\,\mathrm{adj}(\boldsymbol{A})$$

(c)

- Method 1: From part (b), we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boldsymbol{A}^{-1}\begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

- Method 2 (Gauss-Jordan Method):

$$\left[\ \boldsymbol{A} \mid \boldsymbol{b}\ \right] \longrightarrow \left[\ \boldsymbol{I} \mid \boldsymbol{A}^{-1}\boldsymbol{b}\ \right]$$

- Method 3 (Cramer's Rule):

$$D_1 = \det\begin{bmatrix} 9 & -2 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & -2 \end{bmatrix} = -54 \qquad \Longrightarrow \qquad x_1 = \frac{-54}{-27} = 2$$

$$D_2 = \det\begin{bmatrix} 2 & 9 & 1 \\ 1 & 0 & 2 \\ 2 & 0 & -2 \end{bmatrix} = 54 \qquad \Longrightarrow \qquad x_2 = \frac{54}{-27} = -2$$

$$D_3 = \det\begin{bmatrix} 2 & -2 & 9 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} = -27 \qquad \Longrightarrow \qquad x_3 = \frac{-27}{-27} = 1$$

Question 4: (16 points) Find the eigenvalues and the corresponding eigen-vectors of the following 3×3 matrix,

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Answer: The characteristic polynomial is

$$\lambda(\lambda - 1)(\lambda - 2) = \lambda^3 - 3\lambda^2 + 2\lambda.$$

The eigenvalues are 0, 1 and 2.

The eigenvectors corresponding to 0 are $k \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ for $k \neq 0$.

The eigenvectors corresponding to 1 are $k \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ for $k \neq 0$.

The eigenvectors corresponding to 2 are $k \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ for $k \neq 0$.

Question 5

Given a 2x2 matrix $\mathbf{A}$ with eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ and corresponding eigenvectors $\mathbf{e}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

a) Determine the matrix $\mathbf{A}$. (6 pts)

b) Given another matrix $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$. It is known that $\mathbf{B}$ has the same eigenvalues as matrix $\mathbf{A}$. Determine the corresponding eigenvectors for $\mathbf{B}$. (6 pts)

c) Are matrices $\mathbf{A}$ and $\mathbf{B}$ similar?

If yes, describe how you would obtain a similarity transformation matrix $\mathbf{T}$ such that $\mathbf{T}^{-1}\mathbf{AT} = \mathbf{B}$. (8 pts)

Answer:

a) $\mathbf{P} = [\mathbf{e}_1 \quad \mathbf{e}_2] = \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$ diagonalizes matrix $\mathbf{A}$ by $\mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then

$$\mathbf{A} = \mathbf{P}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\mathbf{P}^{-1} = \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}.$$

b) For $\lambda_1 = 2$, eigenvector is $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, For $\lambda_2 = 3$, eigenvector is $\mathbf{e}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

c) Yes, $\mathbf{A}$ and $\mathbf{B}$ are similar, because they are both similar to the diagonal matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

The matrix $\mathbf{Q} = [\mathbf{e}_1 \quad \mathbf{e}_2] = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ diagonalizes matrix $\mathbf{B}$ by $\mathbf{Q}^{-1}\mathbf{B}\mathbf{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

Hence $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{Q}^{-1}\mathbf{B}\mathbf{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow (\mathbf{Q}\mathbf{P}^{-1})\mathbf{A}(\mathbf{P}\mathbf{Q}^{-1}) = \mathbf{B}$, and hence the matrix $\mathbf{T}$ is given by $\mathbf{T} = \mathbf{P}\mathbf{Q}^{-1}$. With $\mathbf{Q}^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$, $\mathbf{T} = \begin{bmatrix} 7 & -2 \\ 10 & -3 \end{bmatrix}$.

Question 6. The *trace* of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ is defined to be the sum of the entries lying on the main diagonal of $\mathbf{A}$. That is, $trace(\mathbf{A}) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{k=1}^{n} a_{kk}$. Prove the following statements:

(1) For any $m \times n$ matrix $\mathbf{B}$ and $n \times m$ matrix $\mathbf{C}$, $trace(\mathbf{BC}) = trace(\mathbf{CB})$. [8pts]

(2) Suppose that $\mathbf{B}$ is an $m \times n$ matrix. If $trace(\mathbf{B}^T\mathbf{B}) = 0$, then $\mathbf{B}$ must be the zero matrix (i.e., $\mathbf{B} = \mathbf{0}$). [8pts]

Solution.

(1) Note that the matrix $\mathbf{BC}$ is of size $m \times m$ and the matrix $\mathbf{CB}$ is of size $n \times n$. Suppose that $\mathbf{B} = [b_{ij}]$ and $\mathbf{C} = [c_{ij}]$. Then, for $k = 1, 2, \ldots, m$,

$$(\mathbf{BC})_{kk} = \sum_{l=1}^{n} b_{kl} c_{lk}.$$

Then, we have

$$trace(\mathbf{BC}) = \sum_{k=1}^{m} \sum_{l=1}^{n} b_{kl} c_{lk}. \tag{1}$$

Similarly, for $k = 1, 2, \ldots, n$,

$$(\mathbf{CB})_{kk} = \sum_{l=1}^{m} c_{kl} b_{lk}.$$

Then, we have

$$trace(\mathbf{CB}) = \sum_{k=1}^{n} \sum_{l=1}^{m} c_{kl} b_{lk}. \tag{2}$$

Since the right hand sides of equalities (1) and (2) are the same, we conclude that $trace(\mathbf{BC}) = trace(\mathbf{CB})$.

(2) Note that the matrix $\mathbf{B}^T\mathbf{B}$ is of size $n \times n$. For $k = 1, 2, \ldots, n$, we have

$$(\mathbf{B}^T\mathbf{B})_{kk} = \sum_{l=1}^{m} (\mathbf{B}^T)_{kl} B_{lk} = \sum_{l=1}^{m} B_{lk}^2.$$

Then, we have

$$trace(\mathbf{B}^T\mathbf{B}) = \sum_{k=1}^{n} \sum_{l=1}^{m} B_{lk}^2.$$

Since $trace(\mathbf{B}^T\mathbf{B}) = 0$, we have

$$\sum_{k=1}^{n} \sum_{l=1}^{m} B_{lk}^2 = 0.$$

Then, we must have $B_{lk} = 0$ for all $l = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$. Thus, $\mathbf{B}$ must be a zero matrix.