Lecture Notes: Curves and Tangent Vectors

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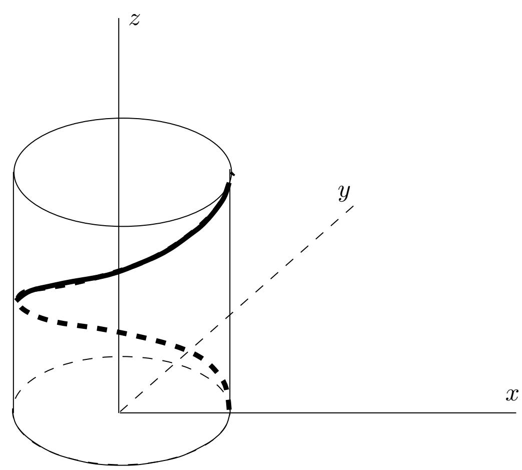
1 Curves

Imagine that you move a point around in $\mathbb{R}^d$. The locus of the point forms a *curve*. Intuitively, a curve is a one-dimensional "in nature": we can represent a curve using a vector function $\boldsymbol{r}(t)$:

$$\boldsymbol{r}(t) = [x_1(t), x_2(t), ..., x_d(t)]$$

where t is a real value in a certain range, and each function $x_i(t)$ (with $i \in [1, d]$) returns a real value. For each t, $(x_1(t), ..., x_d(t))$ defines a point, and $\boldsymbol{r}(t)$ gives the corresponding vector.

For example, $\boldsymbol{r}(t) = [\cos t, \sin t]$ for $t \in [0, 2\pi)$ defines a circle in $\mathbb{R}^2$, whereas $\boldsymbol{r}(t) = [\cos t, \sin t, t]$ for $t \in [0, 2\pi)$ defines a circular helix in $\mathbb{R}^3$ as shown below:



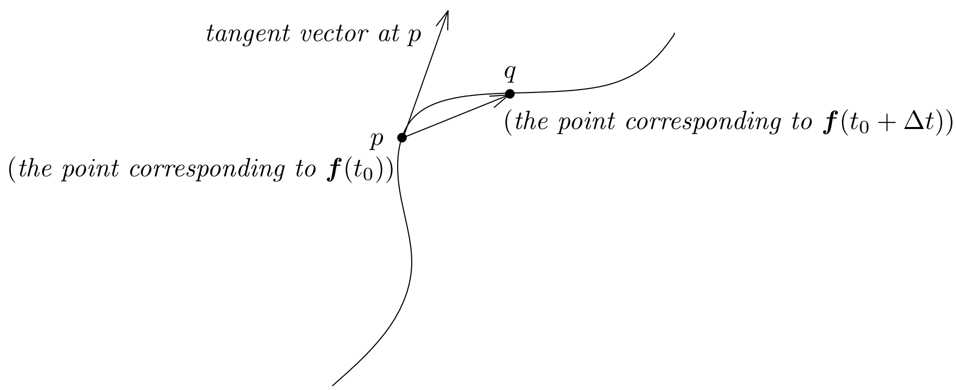
As yet another example, given constant d-dimensional vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ with $\boldsymbol{q} \neq \boldsymbol{0}$, function $\boldsymbol{r}(t) = \boldsymbol{p} + t\boldsymbol{q}$ for $t \in (-\infty, \infty)$ gives a line in $\mathbb{R}^d$.

2 Tangent Vectors

We are ready to introduce:

Definition 1. *Let $\mathbf{r}(t)$ be a curve, t_0 be a value of t, and p be the point corresponding to $\mathbf{r}(t_0)$. If $\boldsymbol{r}(t)$ is differentiable at t_0, then the vector $\mathbf{r}'(t_0)$ is the* **tangent vector** *of the curve at p.*

The tangent vector has an intuitive geometric interpretation. Let q be the point that corresponds to $\boldsymbol{f}(t_0 + \Delta t)$; see the figure below. Let us focus on the direction of the directed segment $\overrightarrow{pq}$. Now, imagine q moving along the curve towards p (namely, Δt tends to 0). The direction of the directed segment gradually converges to the direction of the tangent vector at p.



We will refer to

$$\boldsymbol{u}(t_0) \quad = \quad \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}$$

as the *unit tangent vector* of the curve at p. Note that $|\boldsymbol{u}(t_0)| = 1$.

As an example, consider the helix mentioned earlier: $\boldsymbol{r}(t) = [\cos t, \sin t, t]$ for $t \in [0, 2\pi)$. Let p be the point corresponding to $\boldsymbol{r}(1)$. Then, the tangent vector of the curve at p is $\boldsymbol{r}'(1) = [-\sin(1), \cos(1), 1]$. The unit tangent vector at p is therefore $[-\frac{\sin(1)}{\sqrt{2}}, \frac{\cos(1)}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$.