Lecture Notes: Eigenvalues and Eigenvectors

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1 Definitions

Let $\boldsymbol{A}$ be an $n \times n$ matrix. If there exist a real value λ and a non-zero $n \times 1$ vector $\boldsymbol{x}$ satisfying

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x} \tag{1}$$

then we refer to λ as an *eigenvalue* of $\boldsymbol{A}$, and $\boldsymbol{x}$ as an *eigenvector* of $\boldsymbol{A}$ *corresponding to* λ.

Example 1. Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

You can easily verify that

$$\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Hence, 3 is an eigenvalue of $\boldsymbol{A}$. Vector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector of $\boldsymbol{A}$ corresponding to 3. $\square$

2 Finding All Eigenvalues

Moving $\lambda\boldsymbol{x}$ in (1) to the left hand side gives:

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$$

where $\boldsymbol{I}$ is the $n \times n$ identity matrix. Introducing $\boldsymbol{B} = \boldsymbol{A} - \lambda\boldsymbol{I}$, we can re-write the above as

$$\boldsymbol{B}\boldsymbol{x} = \boldsymbol{0} \tag{2}$$

Let us consider the above as an equation about $\boldsymbol{x}$. We know that if $det(\boldsymbol{B}) \neq 0$, the above equation has a unique solution $\boldsymbol{x} = \boldsymbol{0}$. However, this is *not* what we want. Remember that our goal is to find an eigenvector $\boldsymbol{x}$ of $\boldsymbol{A}$, which needs to be a non-zero vector. Therefore, we must choose λ appropriately to make $det(\boldsymbol{B}) = 0$. This provides us a way to find the eigenvalues of $\boldsymbol{A}$.

Example 2. Consider the matrix $\boldsymbol{A}$ in Example 1.

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{A} - \lambda\boldsymbol{I} \\ &= \begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix} \end{aligned}$$

Hence:

$$\begin{aligned} det(\boldsymbol{B}) &= (1-\lambda)(4-\lambda)+2 \\ &= \lambda^2 - 5\lambda + 6. \end{aligned}$$

To make $det(\boldsymbol{B}) = 0$, we can set λ to $\lambda_1 = 3$ and $\lambda_2 = 2$. These are *all* the eigenvalues of $\boldsymbol{A}$. □

In general, $det(\boldsymbol{B}) = det(\boldsymbol{A} - \lambda \boldsymbol{I})$ is a polynomial function of λ. We refer to the function as the *characteristic polynomial* of $\boldsymbol{A}$. For instance, in Example 2, the characteristic polynomial of $\boldsymbol{A}$ is $\lambda^2 - 5\lambda + 6$. The eigenvalues of $\boldsymbol{A}$ are precisely the solutions of λ in

$$det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0. \tag{3}$$

The above equation is called the *characteristic equation* of $\boldsymbol{A}$.

Lemma 1. *An $n \times n$ matrix $\boldsymbol{A}$ can have at most n distinct eigenvalues.*

Proof. The characteristic polynomial of $\boldsymbol{A}$ is a polynomial of degree n. Hence, Equation (3) can have at most n distinct roots of λ. □

3 Finding All Eigenvectors

Let λ be a value satisfying (3), namely, λ is an eigenvalue of $\boldsymbol{A}$. In this case, Equation (2) has *infinitely many* solutions $\boldsymbol{x}$ (because $det(\boldsymbol{B}) = 0$); we denote by $EigenSpace(\lambda)$ the set of all those solutions $\boldsymbol{x}$. The eigenvectors of $\boldsymbol{A}$ corresponding to λ are exactly the non-zero vectors in $EigenSpace(\lambda)$.

Example 3. Consider again the matrix $\boldsymbol{A}$ in Example 1. We know from Example 2 that it has two eigenvalues: $\lambda_1 = 3$ and $\lambda_2 = 2$.

Let us first look for the eigenvectors of $\boldsymbol{A}$ for $\lambda_1 = 3$. Namely, we want to find $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to satisfy:

$$\begin{aligned} (\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{x} &= \boldsymbol{0} \Rightarrow \\ \begin{bmatrix} 1-3 & -1 \\ 2 & 4-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \\ \begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Hence, any $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ satisfying $-2x_1 - x_2 = 0$ is a solution to the above system. The set of such vectors can be represented in a parametric form: $x_1 = t$ and $x_2 = -2t$ for any $t \in \mathbb{R}$. This set—which we denote as $EigenSpace(\lambda_1)$—has dimension 1. Every non-zero vector in $EigenSpace(\lambda_1)$ is an eigenvector corresponding to λ_1.

Similarly, to obtain the eigenvectors of $\boldsymbol{A}$ for $\lambda_2 = 2$, we want $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to satisfy:

$$\begin{aligned}
(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{x} &= \boldsymbol{0} \Rightarrow \\
\begin{bmatrix} 1-2 & -1 \\ 2 & 4-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \\
\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

Hence, any $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ satisfying $-x_1 - x_2 = 0$ is a solution to the above system. The set of such vectors can be represented in a parametric form: $x_1 = t$ and $x_2 = -t$ for any $t \in \mathbb{R}$. This set, denoted as $EigenSpace(\lambda_2)$, also has dimension 1. Every non-zero vector in $EigenSpace(\lambda_2)$ is an eigenvector corresponding to λ_2. □

Example 4. Consider

$$\boldsymbol{A} = \begin{bmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -3 & -6 & 1 \end{bmatrix}$$

Its characteristic equation is:

$$\begin{aligned}
det(\boldsymbol{A} - \lambda \boldsymbol{I}) &= 0 \Rightarrow \\
\begin{vmatrix} 4-\lambda & 6 & 0 \\ -3 & -5-\lambda & 0 \\ -3 & -6 & 1-\lambda \end{vmatrix} &= 0 \Rightarrow \\
(1-\lambda)\begin{vmatrix} 4-\lambda & 6 \\ -3 & -5-\lambda \end{vmatrix} &= 0 \Rightarrow \\
(1-\lambda)((4-\lambda)(-5-\lambda)+18) &= 0 \Rightarrow \\
(\lambda-1)^2(\lambda+2) &= 0.
\end{aligned}$$

Hence, $\boldsymbol{A}$ has two eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = -2$.

To look for the eigenvectors of $\boldsymbol{A}$ for $\lambda_1 = 1$, we seek $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying

$$\begin{aligned}
(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{x} &= \boldsymbol{0} \Rightarrow \\
\begin{bmatrix} 3 & 6 & 0 \\ -3 & -6 & 0 \\ -3 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \boldsymbol{0} \Rightarrow \\
\begin{bmatrix} 3 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \boldsymbol{0} \Rightarrow
\end{aligned}$$

Hence, any $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying $3x_1 + 6x_2 = 0$ is a solution to the above system. The set of such vectors can be represented in a parametric form: $x_1 = 2u$, $x_2 = -u$, and $x_3 = v$ for any

$(u, v) \in \mathbb{R}^2$. The set is denoted as $\mathit{EigenSpace}(\lambda_1)$, and has dimension 2. Every non-zero vector in $\mathit{EigenSpace}(\lambda_1)$ is an eigenvector corresponding to λ_1.

To look for the eigenvectors of $\boldsymbol{A}$ for $\lambda_2 = -2$, we seek $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying

$$\begin{aligned}
(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{x} &= \boldsymbol{0} \Rightarrow \\
\begin{bmatrix} 6 & 6 & 0 \\ -3 & -3 & 0 \\ -3 & -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \boldsymbol{0} \Rightarrow \\
\begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \boldsymbol{0} \Rightarrow
\end{aligned}$$

Hence, any $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying

$$\begin{aligned}
-x_1 - x_2 &= 0 \\
x_2 - x_3 &= 0
\end{aligned}$$

is a solution to the above system. The set of such vectors can be represented in a parametric form: $x_1 = -t$, $x_2 = t$, and $x_3 = t$ for any $t \in \mathbb{R}$. This set is denoted as $\mathit{EigenSpace}(\lambda_2)$, and has dimension 1. Every non-zero vector in $\mathit{EigenSpace}(\lambda_2)$ is an eigenvector corresponding to λ_2. $\square$

The vector space $\mathit{EigenSpace}(\lambda)$ is referred to as the *eigenspace* of the eigenvalue λ. The dimension of $\mathit{EigenSpace}(\lambda)$ is referred to as the *geometric multiplicity* of λ.

Appendix: Algebraic Multiplicity of Eigenvalues (Not Required by the Syllabus)

Recall that the eigenvalues of an $n \times n$ matrix $\boldsymbol{A}$ are solutions to the characteristic equation (3) of $\boldsymbol{A}$. Sometimes, the equation may have less than n distinct roots, because some roots may happen to be the same. In general, if $\lambda_1, \lambda_2, ..., \lambda_k$ are the distinct roots, then we must be able to re-write equation (3) as:

$$(\lambda - \lambda_1)^{t_1} (\lambda - \lambda_2)^{t_2} ... (\lambda - \lambda_k)^{t_k} = 0$$

where $t_1, t_2, ..., t_k$ are positive integers satisfying $\sum_{i=1}^{k} t_i = n$. We refer to t_i as the *algebraic multiplicity* of λ_i, for each $i \in [1, k]$. It is worth mentioning that some of these roots can be complex numbers, although in this course we will focus on matrices with only real-valued eigenvalues.

Example 5.

- In Example 2, the algebraic multiplicities of the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$ are both 1.
- In Example 4, the algebraic multiplicity of the eigenvalue $\lambda_1 = 1$ is 2, and that of the eigenvalue $\lambda_2 = -2$ is 1.

$\square$

We state the next result without proof:

Lemma 2. *The geometric multiplicity of an eigenvalue is at most its algebraic multiplicity.*

In all the examples we have seen, the geometric multiplicity has always been the same as its algebraic multiplicity. However, this is not always true, as you can see from the example below.

Example 6. Consider

$$\boldsymbol{A} = \begin{bmatrix} -1 & 1 & 0 \\ -4 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Its characteristic equation is

$$\begin{aligned} \begin{vmatrix} -1-\lambda & 1 & 0 \\ -4 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} &= 0 \Rightarrow \\ (\lambda-1)^2(\lambda-2) &= 0. \end{aligned}$$

Hence, $\boldsymbol{A}$ has only 2 eigenvalues: $\lambda_1 = 1$ and $\lambda_2 = 2$. They have algebraic multiplicities 2 and 1, respectively.

Let us now focus on the eigenvalue $\lambda_1 = 1$. We will see that its geometric multiplicity is 1. To find its corresponding eigenvectors $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, we solve the equation:

$$\begin{aligned} (\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{x} &= \boldsymbol{0} \Rightarrow \\ \begin{bmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \boldsymbol{0} \Rightarrow \\ \begin{bmatrix} -2 & 1 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \boldsymbol{0} \Rightarrow \end{aligned}$$

Hence, we know that

$$\begin{aligned} -2x_1 + x_2 &= 0 \\ (1/2)x_2 + x_3 &= 0 \end{aligned}$$

Therefore, the set of solutions to the above problem is the set $\mathit{EigenSpace}(\lambda_1)$ of vectors $\begin{bmatrix} -t \\ -2t \\ t \end{bmatrix}$ for all $t \in \mathbb{R}$. This set has dimension 1. $\square$