SQ7

(a)

$$\int_{-\infty}^{\infty} f^* \frac{d}{dx} g \, dx = f^* g \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g \frac{d}{dx} f^* \, dx$$

$$= - \int_{-\infty}^{\infty} g \frac{d}{dx} f^* \, dx$$

(f & g vanish at infinity)

$$= - \int_{-\infty}^{\infty} \left(\frac{df}{dx}\right)^* g \, dx$$

$\therefore \frac{d}{dx}$ is not a Hermitian operator

(b)

$$\int_{-\infty}^{\infty} f^* \frac{d^2}{dx^2} g \, dx = f^* \frac{dg}{dx} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{dg}{dx} \frac{df^*}{dx} \, dx$$

$$= - \int_{-\infty}^{\infty} \frac{dg}{dx} \frac{df^*}{dx} \, dx$$

$$= - g \frac{df^*}{dx} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} g \frac{d^2 f^*}{dx^2} \, dx$$

$$= \int_{-\infty}^{\infty} g \frac{d^2 f^*}{dx^2} \, dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{d^2 f}{dx^2}\right)^* g \, dx$$

$\therefore \frac{d^2}{dx^2}$ is a Hermitian operator

(c) $\int_{-\infty}^{\infty} f^* x \frac{\hbar}{i}\frac{d}{dx} g \, dx$

$= \frac{\hbar}{i}\left[f^* x g \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} g \frac{d}{dx}(f^* x)\, dx \right]$

$= -\frac{\hbar}{i}\int_{-\infty}^{\infty} g \frac{d}{dx}(x f^*)\, dx$

$= -\frac{\hbar}{i}\int_{-\infty}^{\infty} g f^* \, dx \underbrace{- \frac{\hbar}{i}\int_{-\infty}^{\infty} g x \frac{df^*}{dx}\, dx}_{\text{If } xp_x \text{ is hermitian, we only have this term}}$

$\therefore \hat{X}\hat{P}_x$ is not Hermitian

$\int_{-\infty}^{\infty} f^* \frac{\hbar}{i}\frac{d}{dx}(xg)\, dx$

$= \frac{\hbar}{i}\left[f^* x g \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x g \frac{df^*}{dx}\, dx \right]$

$= -\frac{\hbar}{i}\int_{-\infty}^{\infty} x g \frac{df^*}{dx}\, dx$

$\neq -\int_{-\infty}^{\infty} g \frac{\hbar}{i}\frac{d}{dx}(x f^*)\, dx$

$\therefore \hat{P}_x\hat{X}$ is not Hermitian.

(c) $\int_{-\infty}^{\infty} f^* (\hat{x}\hat{P}_x + \hat{P}_x\hat{x})\, g\, dx$

$$= -\frac{\hbar}{i}\int_{-\infty}^{\infty} \left[g\frac{d}{dx}(xf^*) + xg\frac{df^*}{dx} \right] dx$$

$$= \int_{-\infty}^{\infty} g\, (\hat{x}\hat{P}_x + \hat{P}_x\hat{x})^* f^*\, dx$$

$\therefore\ \hat{x}\hat{P}_x + \hat{P}_x\hat{x}$ is hermitian.

$$i(\hat{x}\hat{p}_x - \hat{P}_x\hat{x}) = i[\hat{x}, \hat{P}_x] = -\hbar$$

$\therefore\ i(\hat{x}\hat{p}_x - \hat{P}_x\hat{x})$ is hermitian

(d) $\int_{-\infty}^{\infty} f^* (\hat{C} + \hat{D}) g \, dx$

$$= \int_{-\infty}^{\infty} f^* \hat{C} g \, dx + \int_{-\infty}^{\infty} f^* \hat{D} g \, dx$$

$$= \int_{-\infty}^{\infty} g \hat{C}^* f^* dx + \int_{-\infty}^{\infty} g \hat{D}^* f^* dx$$

$\hat{C}$ & $\hat{D}$ are Hermitian operator

$$= \int_{-\infty}^{\infty} g (\hat{C}^* + \hat{D}^*) f^* dx$$

$$= \int_{-\infty}^{\infty} g [(\hat{C} + \hat{D}) f]^* dx$$

$\therefore$ $\hat{C} + \hat{D}$ is Hermitian

$\int_{-\infty}^{\infty} f^* (\hat{C} + i\hat{D}) g \, dx$

$$= \int_{-\infty}^{\infty} f^* \hat{C} g \, dx + i \int_{-\infty}^{\infty} f^* \hat{D} g \, dx$$

$$= \int_{-\infty}^{\infty} g \hat{C}^* f^* dx + i \int_{-\infty}^{\infty} g \hat{D}^* f^* dx$$

$$= \int_{-\infty}^{\infty} g (\hat{C}^* + i\hat{D}^*) f^* dx$$

$$\neq \int_{-\infty}^{\infty} g (\hat{C} + i\hat{D})^* f^* dx$$

$\therefore$ $\hat{C} + i\hat{D}$ is not Hermitian

SQ8.

(a) Recall the $l = 1$ hydrogen atom angular part wavefunction:

$$Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta\,; \quad Y_{1,1}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi},$$

and $\hat{L}_x$ expressed as derivatives:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right).$$

Then we evaluate the integral:

$$\int Y_{1,1}^*(\theta,\phi)\hat{L}_x Y_{1,0}(\theta,\phi)d\Omega$$

$$= -\frac{3i\hbar}{4\sqrt{2}\pi}\int \sin\theta\, e^{-i\phi}\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)\cos\theta\, d\Omega$$

$$= \frac{3i\hbar}{4\sqrt{2}\pi}\int \sin\theta\, e^{-i\phi}\sin\phi\sin\theta\,(\sin\theta\, d\theta d\phi)$$

$$= \frac{3i\hbar}{4\sqrt{2}\pi}\int_0^{\pi}\sin^3\theta d\theta \int_0^{2\pi} e^{-i\phi}\sin\phi\, d\phi$$

$$= -\frac{3i\hbar}{4\sqrt{2}\pi}\left(\frac{4}{3}\right)(-i\pi)$$

$$= \frac{\hbar}{\sqrt{2}}$$

whereby $[L_x]_{12}$ (the "12" element of the matrix $[L_x]$) is $\hbar/\sqrt{2}$. And the other elements of $[L_x]$ can be worked out in a similar way.

(b) Recall that to determine if a matrix M is Hermitian or not, we can inspect whether the matrix elements satisfy $M_{ij} = M_{ji}^*$. Consider $[L_x]$:

$$[L_x] = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

which obviously satisfies $M_{ij} = M_{ji}^*$. Hence $[L_x]$ is a Hermitian matrix.

Consider $[L_y]$:

$$[L_y] = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix},$$

which also satisfies $M_{ij} = M_{ji}^*$. Hence $[L_y]$ is a Hermitian matrix.

(c) We write down the matrix $[L_+]$:

$$[L_+] = [L_x] + i[L_y] = \frac{\hbar}{\sqrt{2}}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix} + \frac{i\hbar}{\sqrt{2}}\begin{bmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{bmatrix} = \sqrt{2}\hbar\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}.$$

which does not satisfy $M_{ij} = M_{ji}^*$. Hence $[L_+]$ is not a Hermitian matrix.

We write down the matrix $[L_-]$:

$$[L_-] = [L_x] - i[L_y] = \frac{\hbar}{\sqrt{2}}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix} - \frac{i\hbar}{\sqrt{2}}\begin{bmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{bmatrix} = \sqrt{2}\hbar\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}.$$

which does not satisfy $M_{ij} = M_{ji}^*$. Hence $[L_-]$ is not a Hermitian matrix.

(d) We act $[L_+]$ on the column vector $(0,1,0)^T$:

$$[L_+]\begin{bmatrix}0\\1\\0\end{bmatrix} = \sqrt{2}\hbar\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \sqrt{2}\hbar\begin{bmatrix}1\\0\\0\end{bmatrix}$$

where $(1,0,0)^T$ corresponds to the state $|1,1\rangle$ (i.e. $|l = 1, m = 1\rangle$). The operator $[L_+]$ promotes the state $|1,0\rangle$ into $|1,1\rangle$:

$$\hat{L}_+|1,0\rangle = \sqrt{2}\hbar|1,1\rangle.$$

We act $[L_-]$ on the column vector $(0,1,0)^T$:

$$[L_+]\begin{bmatrix}0\\1\\0\end{bmatrix} = \sqrt{2}\hbar\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \sqrt{2}\hbar\begin{bmatrix}0\\0\\1\end{bmatrix}$$

where $(0,0,1)^T$ corresponds to the state $|1,-1\rangle$. The operator $[L_-]$ reduces the state $|1,0\rangle$ into $|1,-1\rangle$:

$$\hat{L}_-|1,0\rangle = \sqrt{2}\hbar|1,-1\rangle.$$

We see that the operators $\hat{L}_+$ and $\hat{L}_-$ are similar to the ladder operators in a harmonic oscillator problem.

(e) We express $[L_+][L_-]$ as the following:

$$[L_+][L_-] = 2\hbar^2\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix} = 2\hbar^2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix},$$

and

$$[L^2] = 2\hbar^2\boldsymbol{I} = 2\hbar^2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$[L_z^2] = \hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We find that the “something” should be:

$$2\hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - 2\hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \hbar^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \hbar[L_z].$$

Hence the relation is written as:

$$[L_+][L_-] = [L^2] - [L_z^2] + \hbar[L_z].$$