SQ25

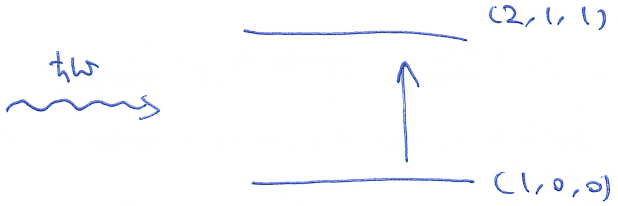
In SQ24, we have already discussed the transition from

$$2p \longleftrightarrow 1s$$

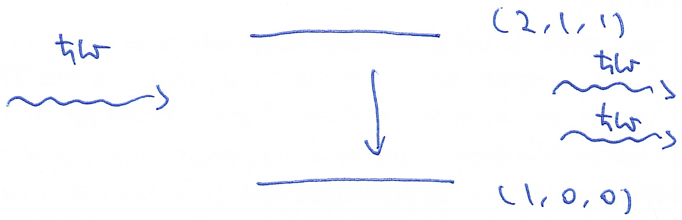
$$(n=2,\ \ell=1,\ m_\ell=\pm1) \qquad (n=1,\ \ell=0,\ m_\ell=0)$$

There are 3 possible ways of transitions:

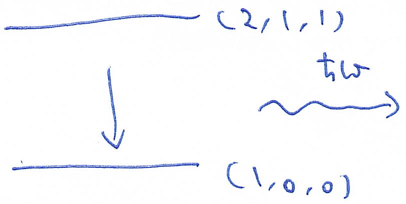
① Stimulated Absorption (1s → 2p) , B_{12}



② Stimulated Emission (2p → 1s) , B_{21}



③ Spontaneous Emission (2p → 1s) , A



In SQ24, we have discussed process ①, and we obtained

$$a_{(2,1,1)}(t) \propto \bar{r}_{(2,1,1),(1,0,0)} = \frac{128}{243} a_0 (\hat{x} - i\hat{y})$$

(Reminder: $\bar{r}_{(2,1,1),(1,0,0)}$ is a complex vector)

a. Next, we move to process ②.

An important note is : Both process ① and ② have the same perturbation, a stimulation of photons " $\overset{\hbar\omega}{\rightsquigarrow}$ ".

Therefore, the perturbation term of the Hamiltonian should be the same, ie,

$$\hat{H}' = -\vec{\mu}_{el}\cdot\vec{\mathcal{E}} = \underline{\underline{\vec{r}}}\cdot(e\vec{\mathcal{E}}_0\cos\omega t)$$

∴ We can write down the integral of $\bar{r}_{1s,2p}$ to be:

$$\bar{r}_{1s,2p} = \int \psi^*_{1s}(\vec{r})\,\vec{r}\,\psi_{2p}(\vec{r})\,d^3r$$

Taking the complex conjugate,

$$\bar{r}^*_{1s,2p} = \int [\psi^*_{1s}(\vec{r})]^*\,\vec{r}^*\,\psi^*_{2p}(\vec{r})\,[d^3r]^*$$

Since the components $\vec{r}$ and d^3r are real,

$$\bar{r}^*_{1s,2p} = \int \psi_{1s}(\vec{r})\,\vec{r}\,\psi^*_{2p}(\vec{r})\,d^3r$$

$$= \int \psi^*_{2p}(\vec{r})\,\vec{r}\,\psi_{1s}(\vec{r})\,d^3r$$

$$= \bar{r}_{2p,1s}$$

Plug the result we obtained in SQ24,

$$\bar{r}_{1s,2p} = \bar{r}^*_{2p,1s}$$

$$= \frac{128}{243}a_0(\hat{x} + i\hat{y}) \quad //$$

Note : $\bar{r}_{1s,2p}$ is also a complex vector.

b. To get the transition coefficients B_{21}, we need $|\bar{r}_{1s,2p}|^2$,

$$|\bar{r}_{1s,2p}|^2 = \bar{r}_{1s,2p}^* \cdot \bar{r}_{1s,2p}$$

$$= \left(\frac{128}{243} a_0\right)^2 (\hat{x} - i\hat{y}) \cdot (\hat{x} + i\hat{y})$$

$$= 2\left(\frac{128}{243} a_0\right)^2$$

$$\approx 0.555\, a_0^2 \quad //$$

Note: $|\bar{r}_{2p,1s}|^2 = \left(\frac{128}{243} a_0\right)^2 (\hat{x} + i\hat{y}) \cdot (\hat{x} - i\hat{y})$

$$= |\bar{r}_{1s,2p}|^2$$

∴ The transition coefficients B_{12} (Process ①) is actually the same as B_{21} (Process ②).

Some background to be optional:

Now, we have already known that

$$B \propto |r_{12}|^2,$$

Using the fact that:

In thermal equilibrium, the number of atoms in state 1s and state 2p are maintained such that the rate of "①, stimulated absorption" is the same as the rate of "②, stimulated emission" plus "③, spontaneous emission".

So this is a link between A coefficient & B coefficient.

c. Finally, we get $A = \frac{\omega^3}{3\pi\epsilon_0 c^3 \hbar} e^2 |r_{21}|^2$,

which depends on 3 factors, a bunch of constants $\frac{1}{3\pi\epsilon_0 c^3 \hbar}$, ω^3 dependences, and $e^2 |\bar{r}_{1s,2p}|^2$ dependences.

c. (cont.) (i) Evaluate $|\mu_{1s,2p}|^2 = e^2 |r_{1s,2p}|^2 = e^2 |r_{21}|^2$,

$$e^2 |r_{21}|^2 = e^2 (2) \left(\frac{128}{243}\right)^2 a_0^2$$

Put $e = 1.602 \times 10^{-19}$ C

$a_0 = 5.292 \times 10^{-11}$ m

$e^2 |r_{21}|^2 \approx 3.988 \times 10^{-59}$ $C^2 m^2$ //

and the electric dipole moment $e|r_{21}| \approx 6.32 \times 10^{-30}$ Cm.

(ii) Evaluate ω

ω is definded as the angular frequency of the incoming photon, i.e.,

$$\omega = \omega_{21} = \frac{E_2 - E_1}{\hbar},$$ where E_1, E_2 are the energy levels of 1s and 2p states respectively.

Put $E_1 = -13.6$ eV, $E_2 = -3.4$ eV,

$\omega \approx 1.550 \times 10^{16}$ s^{-1} //

Note: the photon is in ultraviolet region, slightly beyond the visible light region.

d. To compute the half life $\tau = \frac{1}{A}$ of a 2p state atom, we need to compute A:

$$A = \frac{\omega^3}{3\pi \epsilon_0 c^3 \hbar} e^2 |r_{21}|^2$$

Put the values computed in c):

$\omega \approx 1.550 \times 10^{16}$ s^{-1}

$e^2 |r_{21}|^2 \approx 3.988 \times 10^{-59}$ $C^2 m^2$

d. (cont.) and also put the value of constants :

$$\hbar = 6.626 \times 10^{-34} \ J s$$

$$c = 2.998 \times 10^{8} \ m s^{-1}$$

$$\epsilon_0 = 8.854 \times 10^{-12} \ m^{-3} kg^{-1} s^{4} A^{2}$$

A is computed to be :

$$A = 3.132 \times 10^{8} \ s^{-1}$$

∴ The lifetime τ is :

$$\tau = \frac{1}{A} = 3.193 \times 10^{-9} \ s^{-1} \ //$$

Note:

1. This is a typical lifetime for states with allowed electric dipole transitions downward.
2. We computed the value of A just based on the relation to $|r_{2,1}|$, ie, electric dipole mechanism. In fact, this result turns out to be of the same order as the experiments.

SQ 26.

If we only have stimulated absorption and stimulated emission processes between two states 1 and 2, the dynamical equations are:

$$\begin{cases} \frac{dN_1}{dt} = N_1 \cdot B_{12} U(W,T) \\ \frac{dN_2}{dt} = - N_2 B_{21} U(W,T) \end{cases},$$

At equilibrium, the 1→2 transitions and 2→1 transitions are balanced:

$$\frac{dN_1}{dt} = -\frac{dN_2}{dt}$$

$$\Rightarrow N_1 \cdot B_{12} U(W,T) = N_2 \cdot B_{21} U(W,T)$$

$$\Rightarrow N_1 = N_2,$$

where the stimulated absorption and emission rates are equal, i.e. $B_{12} = B_{21}$.

Hence in equilibrium, the number of atoms in two states 1 and 2 are equal, when the spontaneous emission is considered, then $N_1 > N_2$

SQ27

Suppose we achieve $N_2 = 2N_1$ (in non-equilibrium situation) and we use Eq. (2) , which is valid in equilibrium physics , to evaluate the " temperature " of the system.

$$\frac{N_2}{N_1} = \frac{2N_1}{N_1} = e^{-(E_2-E_1)/kT} = e^{-\hbar\omega/kT}$$

$$2 = e^{-2\pi\hbar c/kT(488\,nm)}$$

$$T \approx -42535.2K$$

Negative temperature
Unreasonable results

Therefore, we can't have population inversion for equilibrium case.