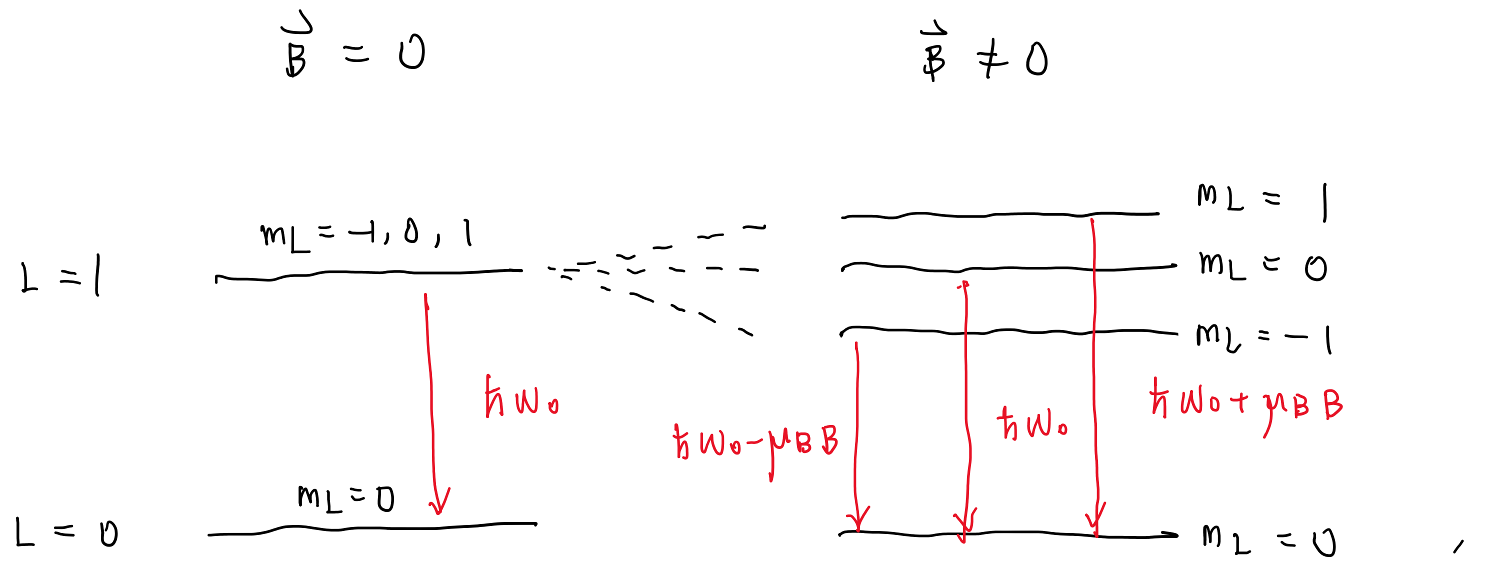
SQ 15,

When a magnetic field is applied, there will be a splitting of spectral lines (spin-effects are ignored), which is called the normal Zeeman effect.

We consider the spectral lines between the s state ($L=0$) and p state ($L=1$) of the helium atom;

(L is the total angular momentum of the atom)

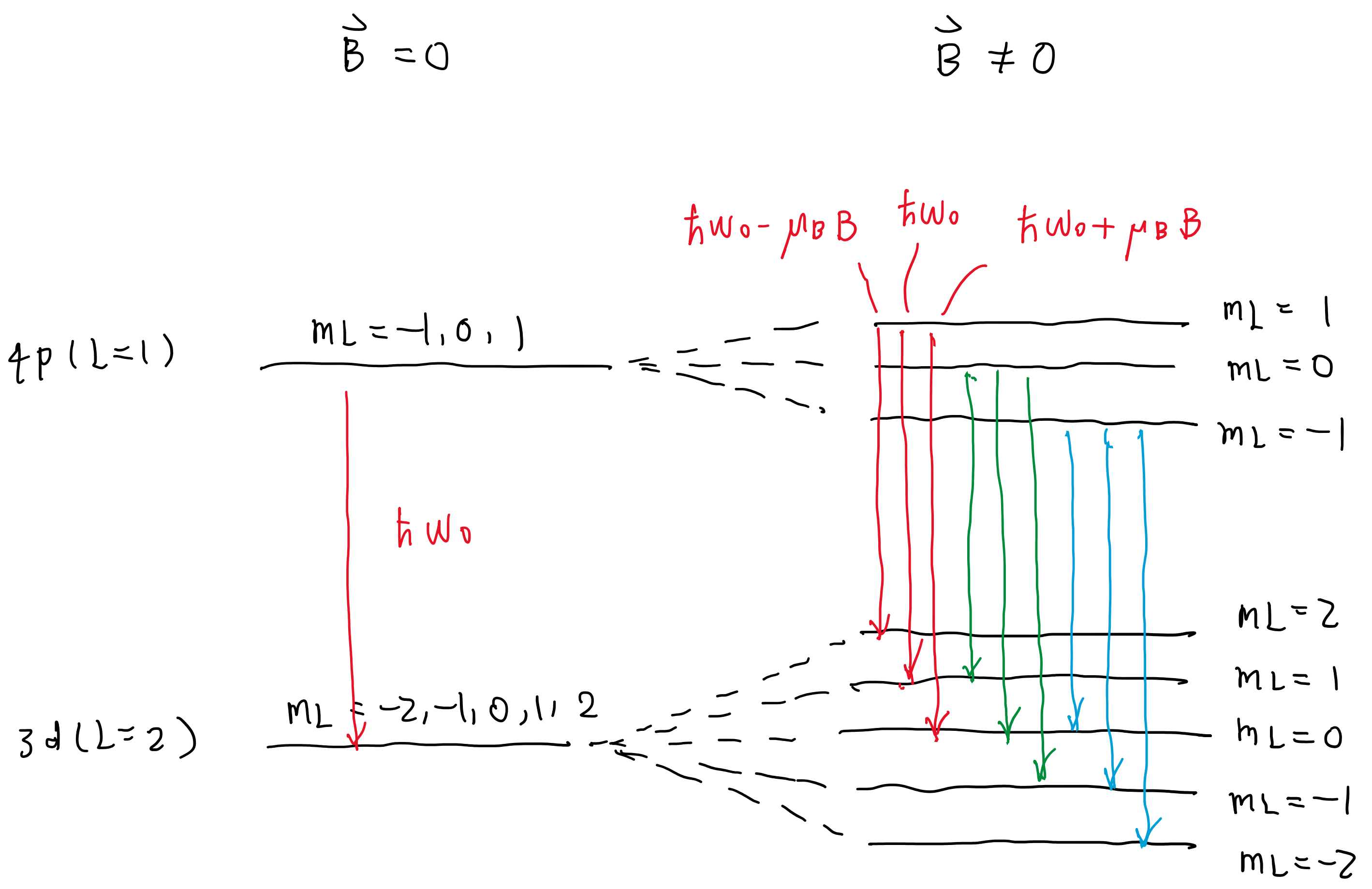


where $\mu_B = \frac{e\hbar}{2m_e}$.

We see one spectral line (ω_0) splits into 3 lines

($\omega_0 - \frac{\mu_B B}{\hbar}$, ω_0, $\omega_0 + \frac{\mu_B B}{\hbar}$)

Then we consider the transitions between an upper p state ($L=1$) to a lower d state ($L=2$), e.g. between 4p and 3d:



According to the selection rule $\Delta m_L = 0, \pm 1$, there are 9 transitions. But the spectral line at $\vec{B}=0$ (ω_0) still spilts into 3 different lines ($\omega_0 - \frac{\mu_B B}{\hbar}$, ω_0, $\omega_0 + \frac{\mu_B B}{\hbar}$) when $\vec{B}\neq 0$.

SQ16

(a) Number of states for (j_1, m_{j_1}) : $2j_1 + 1$

Number of states for (j_2, m_{j_2}) : $2j_2 + 1$

$\therefore$ Number of states for $(j_1, m_{j_1}, j_2, m_{j_2}) = (2j_1+1)(2j_2+1)$

Total angular momentum quantum number ranges from $j_1 - j_2$ to $j_1 + j_2$ in steps of 1 if $j_1 > j_2$. For each j inside this range, we have $(2j+1)$ values of m_j. So the total number of states in this label:

$$\sum_{j=j_1-j_2}^{j_1+j_2} (2j+1) = 2\sum_{j=j_1-j_2}^{j_1+j_2} j + \sum_{j=j_1-j_2}^{j_1+j_2} 1$$

$$= 2\,\frac{2j_2+1}{2}\left[2(j_1-j_2) + 2j_2\right] + 2j_2+1$$

(The first term can be calculated by sum of arithmetic series while the second term just counts the number of j between j_1-j_2 and j_1+j_2)

$$= (2j_2+1)(2j_1-2j_2+2j_2+1)$$

$$= (2j_2+1)(2j_1+1)$$

$\therefore$ Number of states in the two labels are the same.

SQ16

(b) For the labelling scheme $(j_1 = 5/2,\ m_{j_1},\ j_2 = 1,\ m_{j_2})$, we have

$\vert j_1 = 5/2, m_{j_1} = 5/2, j_2 = 1, m_{j_2} = 1\rangle$ $\vert j_1 = 5/2, m_{j_1} = 5/2, j_2 = 1, m_{j_2} = 0\rangle$ $\vert j_1 = 5/2, m_{j_1} = 5/2, j_2 = 1, m_{j_2} = -1\rangle$	$\vert j_1 = 5/2, m_{j_1} = -1/2, j_2 = 1, m_{j_2} = 1\rangle$ $\vert j_1 = 5/2, m_{j_1} = -1/2, j_2 = 1, m_{j_2} = 0\rangle$ $\vert j_1 = 5/2, m_{j_1} = -1/2, j_2 = 1, m_{j_2} = -1\rangle$
$\vert j_1 = 5/2, m_{j_1} = 3/2, j_2 = 1, m_{j_2} = 1\rangle$ $\vert j_1 = 5/2, m_{j_1} = 3/2, j_2 = 1, m_{j_2} = 0\rangle$ $\vert j_1 = 5/2, m_{j_1} = 3/2, j_2 = 1, m_{j_2} = -1\rangle$	$\vert j_1 = 5/2, m_{j_1} = -3/2, j_2 = 1, m_{j_2} = 1\rangle$ $\vert j_1 = 5/2, m_{j_1} = -3/2, j_2 = 1, m_{j_2} = 0\rangle$ $\vert j_1 = 5/2, m_{j_1} = -3/2, j_2 = 1, m_{j_2} = -1\rangle$
$\vert j_1 = 5/2, m_{j_1} = 1/2, j_2 = 1, m_{j_2} = 1\rangle$ $\vert j_1 = 5/2, m_{j_1} = 1/2, j_2 = 1, m_{j_2} = 0\rangle$ $\vert j_1 = 5/2, m_{j_1} = 1/2, j_2 = 1, m_{j_2} = -1\rangle$	$\vert j_1 = 5/2, m_{j_1} = -5/2, j_2 = 1, m_{j_2} = 1\rangle$ $\vert j_1 = 5/2, m_{j_1} = -5/2, j_2 = 1, m_{j_2} = 0\rangle$ $\vert j_1 = 5/2, m_{j_1} = -5/2, j_2 = 1, m_{j_2} = -1\rangle$

We get $(2\cdot\frac{5}{2}+1)(2\cdot 1+1) = 18$ states in this labelling

SQ16

(b) For the labelling scheme $(j_1 = 5/2,\ j_2 = 1,\ j,\ m_j)$:

$|j_1 = 5/2, j_2 = 1, j = 7/2, m_j = 7/2\rangle$

$|j_1 = 5/2, j_2 = 1, j = 7/2, m_j = 5/2\rangle$

$|j_1 = 5/2, j_2 = 1, j = 7/2, m_j = 3/2\rangle$

$|j_1 = 5/2, j_2 = 1, j = 7/2, m_j = 1/2\rangle$

$|j_1 = 5/2, j_2 = 1, j = 7/2, m_j = -1/2\rangle$

$|j_1 = 5/2, j_2 = 1, j = 7/2, m_j = -3/2\rangle$

$|j_1 = 5/2, j_2 = 1, j = 7/2, m_j = -5/2\rangle$

$|j_1 = 5/2, j_2 = 1, j = 7/2, m_j = -7/2\rangle$

$|j_1 = 5/2, j_2 = 1, j = 5/2, m_j = 5/2\rangle$

$|j_1 = 5/2, j_2 = 1, j = 5/2, m_j = 3/2\rangle$

$|j_1 = 5/2, j_2 = 1, j = 5/2, m_j = 1/2\rangle$

$|j_1 = 5/2, j_2 = 1, j = 5/2, m_j = -1/2\rangle$

$|j_1 = 5/2, j_2 = 1, j = 5/2, m_j = -3/2\rangle$

$|j_1 = 5/2, j_2 = 1, j = 5/2, m_j = -5/2\rangle$

$|j_1 = 5/2, j_2 = 1, j = 3/2, m_j = 3/2\rangle$

$|j_1 = 5/2, j_2 = 1, j = 3/2, m_j = 1/2\rangle$

$|j_1 = 5/2, j_2 = 1, j = 3/2, m_j = -1/2\rangle$

$|j_1 = 5/2, j_2 = 1, j = 3/2, m_j = -3/2\rangle$

We also have 18 states in this labelling

SQ17

When an external field is introduced, the "minimal substitution rule" is applied such that:

$$H = \frac{\vec{p}^2}{2m} + V(\vec{r}) \longrightarrow H = \frac{(\vec{p} + e\vec{A})^2}{2m} + V(\vec{r}).$$

where $\vec{A}$ is the vector potential. (We stay classical first)

e.g. The external field is:

$$\vec{B} = B\hat{z}$$

we can choose a vector potential $\vec{A}$ such that

$$\vec{\nabla} \times \vec{A} = \vec{B}$$

For example, $\vec{A} = \frac{1}{2}Bx\hat{y} - \frac{1}{2}By\hat{x}$

$$\boxed{\begin{aligned} \therefore \vec{\nabla} \times \vec{A} &= \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}, \\ &= B\hat{z} \end{aligned}}$$

Now, the Hamiltonian becomes:

$$H = \frac{1}{2m}(\vec{p} + e\vec{A})^2 + \underline{V(r)}$$

$\boxed{\text{V(r) is assumed to be spherically symmetric}}$

Expanding H by using $\vec{p} = P_x\hat{x} + P_y\hat{y} + P_z\hat{z}$

and $\vec{A} = -\frac{1}{2}By\hat{x} + \frac{1}{2}Bx\hat{y}$

$$H = \frac{1}{2m}\left((P_x - \frac{e}{2}By)\hat{x} + (P_y + \frac{e}{2}Bx)\hat{y} + P_z\hat{z}\right)^2 + V(r)$$

$$= \frac{1}{2m}\left[(P_x - \frac{e}{2}By)^2 + (P_y + \frac{e}{2}Bx)^2 + P_z^2\right] + V(r)$$

Expanding the squares,

$$H = \frac{1}{2m}\left[P_x^2 + P_y^2 + P_z^2 - eP_x By + eP_y Bx + \frac{e^2}{4}B^2y^2 + \frac{e^2}{4}B^2x^2 \right] + V(r)$$

$$= \frac{1}{2m}\left[P^2 + eB(xP_y - yP_x) + \frac{e^2}{4}B^2(x^2+y^2)\right] + V(r),$$

where $P^2 = P_x^2 + P_y^2 + P_z^2$,

Note that $xP_y - yP_x$ is the angular momentum L_z, and $\vec{B} = B\hat{z}$

$\therefore$ We can rewrite the 2nd term as follow:

$$eB(xP_y - yP_x) = eBL_z$$
$$= e\vec{B}\cdot\vec{L}.$$

The angular momentum can be related to the electron orbital magnetic momentum by : $\vec{\mu_L} = -\frac{e}{2m}\vec{L}$.

$$\therefore\ eB(xP_y - yP_x) = -2m\,\vec{B}\cdot\vec{\mu_L}$$

The Hamiltonian becomes :

$$H = \frac{P^2}{2m} - \vec{B}\cdot\vec{\mu_L} + \frac{e^2}{8m}B^2(x^2+y^2) + V(r).$$

$\therefore$ There are two extra terms for an introduction of external B field.

① The term $-\vec{\mu_L}\cdot\vec{B}$ is exactly what we got if we apply our EM knowledge on orbital magnetic dipole moment.

② There is another extra term $\frac{e^2}{8m}B^2(x^2+y^2) = \frac{e^2}{2m}A^2 \sim A^2$ as a by-product, which is the diamagnetic response of the orbiting electron.

Not for exam purpose : In EM, we have learned that we can choose any $\vec{A}$ s.t. $\vec{\nabla}\times\vec{A} = \vec{B}$ is fullfilled, and the Physics does not change if we change a different $\vec{A}$ (gauge freedom). Does the H we discussed above change ? If yes, why ? (It is difficult ! Just for fun !)