PHYS3022 APPLIED QUANTUM MECHANICS

SAMPLE QUESTIONS FOR DISCUSSION IN WEEK 3 EXERCISE CLASSES (20-24 January 2020)

What are Sample Questions (SQs)? TA will discuss the **SAMPLE QUESTIONS** in exercise classes. The Sample Questions are designed to serve several purposes. They either review what you have learnt in previous courses, supplement our discussions in lectures, or closed related to the questions in an upcoming Problem. You are encouraged to think about (or work out) the sample questions before attending exercise class and ask the TA questions.

SQ7: Hermitian Operators
SQ8: Operators in matrix form (orbital angular momentum of $\ell = 1$)

SQ7 *Hermitian Operators*

Assume that the functions on which the operators operate are properly well behaved at infinity when working out the following parts.

(a) Is $\frac{d}{dx}$ a Hermitian operator?

(b) Is $\frac{d^2}{dx^2}$ a Hermitian operator?

(c) We know that $\hat{x}$ (position operator) is Hermitian and $\hat{p}_x = \frac{\hbar}{i}\frac{d}{dx}$ is Hermitian. Is the product $\hat{x}\hat{p}_x$ Hermitian? Is the product $\hat{p}_x\hat{x}$ Hermitian? Is $(\hat{x}\hat{p}_x + \hat{p}_x\hat{x})$ Hermitian? Is $i(\hat{x}\hat{p}_x - \hat{p}_x\hat{x})$ Hermitian?

(d) Let $\hat{C}$ and $\hat{D}$ be two Hermitian operators. Is $(\hat{C} + \hat{D})$ Hermitian? Is $(\hat{C} + i\hat{D})$ Hermitian?

SQ8 *The $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$ matrices for $\ell = 1$ orbital angular momentum* (See Problem 1.1.)

In Problem 1.1, the $\ell = 1$ orbital angular momentum components are represented by 3×3 matrices:

$$[L_x] = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{1}$$

$$[L_y] = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{2}$$

$$[L_z] = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3}$$

It follows that the magnitude squared $[L^2]$ is also a 3×3 matrix.

(a) Here, TA will illustrate why $[L_x]_{12}$ (the "12" element of the matrix $[L_x]$) is $\hbar/\sqrt{2}$. Inspecting $[L_z]$, the matrices are constructed by enforcing $[L_z]$ to be a diagonal matrix. Therefore, the basis used in writing the first, second, and third rows and columns are in the order of $Y_{1,1}$, $Y_{1,0}$, and $Y_{1,-1}$. So $[L_x]_{12}$ is related to an integral $\int Y^*_{1,1}(\theta,\phi)\, \hat{L}_x\, Y_{1,0}(\theta,\phi)\, d\Omega$, and $\hat{L}_x$ can be expressed in terms of derivatives with respect to θ and ϕ. Work it out.

(b) Are $[L_x]$ and $[L_y]$ Hermitian matrices?

(c) Construct $[L_\pm] \equiv [L_x] \pm i[L_y]$. Are they Hermitian matrices? [c.f. SQ7]

(d) Consider a column vector $(0, 1, 0)^T$, which is an eigenvector of $[L_z]$. What does $[L_\pm]$ do to it? [Remark: You saw similar results in harmonic oscillator last term.]

(e) Motivated by high school math of $(a+ib)(a-ib) = a^2 + b^2$, it looks as if $[L_+][L_-]$ can be related to $[L_x][L_x] + [L_y][L_y] = [L_x^2] + [L_y^2] = [L^2] - [L_z][L_z] = [L^2] - [L_z^2]$. However, from our experience with similar operators in the harmonic oscillator problem, life is slightly more complicated in quantum mechanics due to the commutator relations. Show that

$$[L_+][L_-] = [L^2] - [L_z^2] + \text{something} \tag{4}$$

and find what that (something) is.