PHYS 5130 Problem Set 5 Solution

1.

Solution: For clarity, each of the five single particle state is referred to by a number.

Energy	Number
0 (First)	1
0 (Second)	2
ϵ (First)	3
ϵ (Second)	4
2ϵ	5

(a) Then, the possible two particles states are as follow (Particle 1, Particle 2): (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5). Each state contributes $e^{-\beta(E_{particle1}+E_{particle2})}$ to the partition function. Performing this summation, one obtains

$$Z_{distinct} = 4 + 8e^{-\beta\epsilon} + 8e^{-2\beta\epsilon} + 4e^{-3\beta\epsilon} + e^{-4\beta\epsilon}. \tag{1}$$

Alternatively, equation 2 in the problem could be used to obtain the partition function. One could first note that the two particles could take on the following total energy $0, \epsilon, 2\epsilon, 3\epsilon, 4\epsilon$. For $E = 0$, both particles have to be in either state 1 or 2. So there are $W = C_1^2 \times C_1^2 = 4$ two particles states with energy $E = 0$. Similarly, for $E = \epsilon$, one particle has to be in either 1 or 2 state, and the other has to be in either 3 or 4 state, moreover, as the two particles are distinguishable, there is an extra factor of 2 ((0, 3), (3, 0) being distinct states). so $W = 2 \times C_1^2 \times C_1^2 = 8$. For $E = 2\epsilon$, either each particle contributes ϵ of energy, or one particle contributes 2ϵ of energy alone. Therefore, $W = C_1^2 \times C_1^2 + C_1^1 \times C_1^2 \times 2 = 8$. For $E = 3\epsilon$, $W = C_1^2 \times C_1^1 \times 2 = 4$. For $E = 4\epsilon$, $W = C_1^1 \times C_1^1 = 1$.

Then, using equation 2 provided in the question, one again obtains

$$Z_{distinct} = 4 + 8e^{-\beta\epsilon} + 8e^{-2\beta\epsilon} + 4e^{-3\beta\epsilon} + e^{-4\beta\epsilon}. \tag{2}$$

For a single particle, the partition function is simply given by

$$\begin{aligned} z_{sp} &= 1 + 1 + e^{-\beta\epsilon} + e^{-\beta\epsilon} + e^{-2\beta\epsilon} && (3) \\ &= 2 + 2e^{-\beta\epsilon} + e^{-2\beta\epsilon}. && (4) \end{aligned}$$

Then,

$$\begin{aligned} z_{sp}^2 &= (2 + 2e^{-\beta\epsilon} + e^{-2\beta\epsilon})^2 && (5) \\ &= 4 + 8e^{-\beta\epsilon} + 8e^{-2\beta\epsilon} + 4e^{-3\beta\epsilon} + e^{-4\beta\epsilon} && (6) \\ &= Z_{distinct}. && (7) \end{aligned}$$

Therefore, $Z_{distinct}$ could be factorized into zs_{sp}^2.

(b) For two identical fermions, the possible two particle states are ((State of one particle, State of the other particle)): (1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5). One then obtains

$$Z_{fermion} = 1 + 4e^{-\beta\epsilon} + 3e^{-2\beta\epsilon} + 2e^{-3\beta\epsilon}. \tag{8}$$

(c) For two identical bosons, the possible two particle states are ((State of one particle, State of another particle)): (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 2), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5), (4, 4), (4, 5), (5, 5). One then obtains

$$Z_{boson} = 3 + 4e^{-\beta\epsilon} + 5e^{-2\beta\epsilon} + 2e^{-3\beta\epsilon} + e^{-4\beta\epsilon}. \tag{9}$$

(d) As Z_{boson} concerns two identical boson, states like (1, 2), (2, 1) in the distinguishable particles case which are considered to be two states are counted only once in the identical boson case.

(e) By direct calculation, one could see that $\frac{1}{2!}Z_{distinct} \neq Z_{boson}$. In distinguishable case, states like (1, 2), (2, 1) (Particle 1, Particle 2) are considered as two different states, but in indistinguishable case, the two are considered the same and counted once only, so such states in distingishable case are overcounted in indistinguishable case. In this case, the result is not correct as states which are not overcounted (Like (1, 1)) are also reduced by a factor of 2.

(f) If now the two particles are not allowed to occupy the same single particle state, one would have to remove states like (1, 1) from the counting, and recompute the partition functions, which are given by

$$Z_{distinct} = 2 + 8e^{-\beta\epsilon} + 6e^{-2\beta\epsilon} + 4e^{-3\beta\epsilon}, \tag{10}$$

$Z_{fermion}$ remains unchanged.

$$Z_{boson} = 1 + 4e^{-\beta\epsilon} + 3e^{-2\beta\epsilon} + 2e^{-3\beta\epsilon}. \tag{11}$$

One could then see that $\frac{1}{2!}Z_{distinct} = Z_{fermion} = Z_{boson}$.

2.

Solution:

(a)

Two particle state (State of one particle, State of the other particle)	Occupation Numbers	Energy
(1, 2)	(1, 1, 0, 0, 0)	0ϵ
(1, 3)	(1, 0, 1, 0, 0)	1ϵ
(1, 4)	(1, 0, 0, 1, 0)	1ϵ
(1, 5)	(1, 0, 0, 0, 1)	2ϵ
(2, 3)	(0, 1, 1, 0, 0)	1ϵ
(2, 4)	(0, 1, 0, 1, 0)	1ϵ
(2, 5)	(0, 1, 0, 0, 1)	2ϵ
(3, 4)	(0, 0, 1, 1, 0)	2ϵ
(3, 5)	(0, 0, 1, 0, 1)	3ϵ
(4, 5)	(0, 0, 0, 1, 1)	3ϵ

So there is 1 state with $E = 0$, 4 states with $E = 1\epsilon$, 3 states with $E = 2\epsilon$, 2 states with $E = 3\epsilon$.
Using equation 2 in the question, one then obtains

$$Z_{fermion} = 1 + 4e^{-\beta\epsilon} + 3e^{-2\beta\epsilon} + 2e^{-3\beta\epsilon} \tag{12}$$

as expected.

(b)

Two particle state (State of one particle, State of the other particle)	Occupation Numbers	Energy
(1, 1)	(2, 0, 0, 0, 0)	0ϵ
(1, 2)	(1, 1, 0, 0, 0)	0ϵ
(1, 3)	(1, 0, 1, 0, 0)	1ϵ
(1, 4)	(1, 0, 0, 1, 0)	1ϵ
(1, 5)	(1, 0, 0, 0, 1)	2ϵ
(2, 2)	(0, 2, 0, 0, 0)	0ϵ
(2, 3)	(0, 1, 1, 0, 0)	1ϵ
(2, 4)	(0, 1, 0, 1, 0)	1ϵ
(2, 5)	(0, 1, 0, 0, 1)	2ϵ
(3, 3)	(0, 0, 2, 0, 0)	2ϵ
(3, 4)	(0, 0, 1, 1, 0)	2ϵ
(3, 5)	(0, 0, 1, 0, 1)	3ϵ
(4, 4)	(0, 0, 0, 2, 0)	2ϵ
(4, 5)	(0, 0, 0, 1, 1)	3ϵ
(5, 5)	(0, 0, 0, 0, 2)	4ϵ

So there are 3 states with $E = 0$, 4 states with $E = \epsilon$, 5 states with $E = 2\epsilon$, 2 states with $E = 3\epsilon$, 1 state with $E = 4\epsilon$. One could then obtain

$$Z_{boson} = 3 + 4e^{-\beta\epsilon} + 5e^{-2\beta\epsilon} + 2e^{-3\beta\epsilon} + e^{-4\beta\epsilon} \tag{13}$$

as expected.

3.

Solution:

(a)

$$
\begin{aligned}
Z &= \sum_{AllStates} e^{-\beta E_{tot}} && (14)\\
&= \underbrace{\sum_{i=1}^{3} e^{-\beta E_i} \dots \sum_{i=1}^{3} e^{-\beta E_i}}_{\text{N}} && (15)\\
&= \prod^{N} \sum_{i=1}^{3} e^{-\beta E_i} && (16)\\
&= z^N && (17)
\end{aligned}
$$

(b)

$$
\begin{aligned}
< E > &= kT^2 \frac{\partial \ln Z}{\partial T} && (18)\\
&= kT^2 N \frac{\partial \ln z}{\partial T} && (19)\\
&= kT^2 N \left(e^{\beta\epsilon} + 1 + e^{-\beta\epsilon}\right)^{-1} \left(-\frac{\epsilon}{kT^2} e^{\beta\epsilon} + \frac{\epsilon}{kT^2} e^{-\beta\epsilon}\right) && (20)\\
&= -N g\mu_B B \frac{e^{\frac{g\mu_B B}{kT}} - e^{-\frac{g\mu_B B}{kT}}}{e^{\frac{g\mu_B B}{kT}} + 1 + e^{-\frac{g\mu_B B}{kT}}} && (21)
\end{aligned}
$$

(c) For each particle, it could be in one of the three states each with probability given by $\frac{e^{-\beta\epsilon_i}}{z}$, where ϵ_i is the energy associated with a single particle state, moreover each state corresponds to a particular value of magnetic dipole moment. So the average dipole moment is given by the sum of possible dipole moment multiplied by its probability, given by

$$
\begin{aligned}
< \mu_z > &= \left(g\mu_B e^{\frac{g\mu_B B}{kT}} + (0)(1) + (-g\mu B) e^{-\frac{g\mu_B B}{kT}}\right) (e^{\frac{g\mu_B B}{kT}} + 1 + e^{-\frac{g\mu_B B}{kT}})^{-1} && (22)\\
&= g\mu_B \left(e^{\frac{g\mu_B B}{kT}} - e^{-\frac{g\mu_B B}{kT}}\right) (e^{\frac{g\mu_B B}{kT}} + 1 + e^{-\frac{g\mu_B B}{kT}})^{-1} && (23)
\end{aligned}
$$

(d)

$$
\begin{aligned}
M &= \frac{N}{V} < \mu_z > && (24)\\
&= \frac{g\mu_B N}{V} \left(e^{\frac{g\mu_B B}{kT}} - e^{-\frac{g\mu_B B}{kT}}\right) (e^{\frac{g\mu_B B}{kT}} + 1 + e^{-\frac{g\mu_B B}{kT}})^{-1} && (25)
\end{aligned}
$$

(e) When the magnetic field strength is high, and temperature is low such that $g\mu_B B >> kT$, $M \rightarrow \frac{Ng\mu_B}{V} \frac{e^{\frac{g\mu_B B}{kT}}}{e^{\frac{g\mu_B B}{kT}}} = \frac{Ng\mu_B}{V}$. On the other hand, when $g\mu_B B << kT$, $e^{\frac{g\mu_B B}{kT}} \approx 1 + \frac{g\mu_B B}{kT}$. Then $M \rightarrow \frac{2N(g\mu_B)^2 B}{3VkT}$. Therefore, susceptibility is inversely proportional to T, which is the Curie's law.

4.

Solution:

(a) Given $D = A\omega$, one could make use of the property that there are only $2N$ oscillation modes in the system to obtain

$$
\begin{aligned}
2N &= \int_0^{\omega_D} A\omega d\omega && (26)\\
&= \frac{1}{2} A\omega_D^2 && (27)
\end{aligned}
$$

Therefore $A = \frac{4N}{\omega_D^2}$.

Moreover,

$$E_{GS} = \frac{1}{2}\hbar \int_0^{\omega_D} D\omega d\omega \tag{28}$$
$$= \frac{2N\hbar}{\omega_D^2} \int_0^{\omega_D} \omega^2 d\omega \tag{29}$$
$$= \frac{2}{3} N\hbar\omega_D \tag{30}$$

So, $\omega_D = \frac{3E_{GS}}{2N\hbar}$.

(b)

$$< E > = E_{GS} + \int_0^{\omega_D} D\frac{\hbar\omega}{e^{\beta\hbar\omega}-1} d\omega \tag{31}$$
$$= E_{GS} + \frac{4N\hbar}{\omega_D^2} \int_0^{\omega_D} \frac{\omega^2}{e^{\beta\hbar\omega}-1} d\omega \tag{32}$$
$$= E_{GS} + \frac{4N}{\omega_D^2 \beta^3 \hbar^2} \int_0^{\beta\hbar\omega_D} \frac{(\beta\hbar\omega)^2}{e^{\beta\hbar\omega}-1} d\beta\hbar\omega \tag{33}$$
$$= E_{GS} + \frac{4Nk^3T^3}{\omega_D^2 \hbar^2} \int_0^{\beta\hbar\omega_D} \frac{x^2}{e^x-1} dx \tag{34}$$

At low temperature, $\beta\hbar\omega_D \to \infty$, so the expression could be approximated by

$$< E > = E_{GS} + \frac{4Nk^3T^3}{\omega_D^2 \hbar^2} \int_0^{\infty} \frac{x^2}{e^x-1} dx \tag{35}$$
$$= E_{GS} + \frac{4Nk^3T^3}{\omega_D^2 \hbar^2}(2.404) \tag{36}$$

As $C = \frac{\partial E}{\partial T}$, one obtains

$$C = E_{GS} + \frac{12Nk^3T^2}{\omega_D^2 \hbar^2}(2.404) \tag{37}$$

So heat capacity is proportional to T^2.

(c) Given that $D \propto \omega^{d-1}$, so $D = A\omega^{d-1}$, where A is a constant to be determined, one could follow the treatment above and obtain

$$< E > = E_{GS} + A\hbar \int_0^{\omega_D} \frac{\omega^d}{e^{\beta\hbar\omega}-1} d\omega \tag{38}$$
$$= E_{GS} + \frac{A}{\beta^{d+1}\hbar^d} \int_0^{\beta\hbar\omega_D} \frac{(\beta\hbar\omega)^d}{e^{\beta\hbar\omega}-1} d\beta\hbar\omega \tag{39}$$
$$= E_{GS} + \frac{Ak^{d+1}T^{d+1}}{\hbar^d} \int_0^{\beta\hbar\omega_D} \frac{x^d}{e^x-1} dx \tag{40}$$

At low temperature, the expression could be approximated by

$$< E >= E_{GS} + \frac{Ak^{d+1}T^{d+1}}{\hbar^d} \int_0^{\infty} \frac{x^d}{e^x-1} dx \tag{41}$$

As $C = \frac{\partial E}{\partial T}$, one could see that $C \propto T^d$.