PHYS 5130 Problem Set 1 Solution

1. **Solution:**

(a) i.

$$z = \cos\left(x^2 + 2y^2\right) \tag{1}$$
$$= \cos\left(t^2 + 2t^4\right) \tag{2}$$

By doing a direct differentiation, one obtains

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\sin\left(t^2 + 2t^4\right)\left(2t + 8t^3\right) \tag{3}$$

ii. In this part, the chain rule is used to compute $\frac{\mathrm{d}z}{\mathrm{d}t}$. By chain rule,

$$\frac{\mathrm{d}z(x,y)}{\mathrm{d}t} = \left(\frac{\partial z}{\partial x}\right)_y \frac{\mathrm{d}x}{\mathrm{d}t} + \left(\frac{\partial z}{\partial y}\right)_x \frac{\mathrm{d}y}{\mathrm{d}t} \tag{4}$$
$$= \left(-2x\sin\left(x^2+2y^2\right)\right) \times 1 + \left(-4y\sin\left(x^2+2y^2\right)\right) \times 2t \tag{5}$$
$$= -\sin\left(t^2 + 2t^4\right)\left(2t + 8t^3\right) \tag{6}$$

(b) Direct differentiation:

$$z = x^2 - xy + y^2 \tag{7}$$
$$= r^2\cos^2\theta - r^2\sin\theta\cos\theta + r^2\sin^2\theta \tag{8}$$
$$= r^2(1 - \sin\theta\cos\theta) \tag{9}$$

$$\left(\frac{\partial z}{\partial r}\right)_\theta = 2r(1 - \sin\theta\cos\theta) \tag{10}$$
$$= r(2 - \sin(2\theta)) \tag{11}$$

$$\left(\frac{\partial z}{\partial \theta}\right)_r = r^2(0 - (\sin\theta \times -\sin\theta + \cos\theta \times \cos\theta)) \tag{12}$$
$$= -r^2(\cos^2\theta - \sin^2\theta) \tag{13}$$
$$= -r^2\cos(2\theta) \tag{14}$$

Chain rule:

$$\left(\frac{\partial z}{\partial r}\right)_\theta = \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial r}\right)_\theta + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial r}\right)_\theta \tag{15}$$
$$= \cos\theta(2x - y) + \sin\theta(-x + 2y) \tag{16}$$
$$= \cos\theta(2r\cos\theta - r\sin\theta) + \sin\theta(-r\cos\theta + 2r\sin\theta) \tag{17}$$
$$= 2r(\cos^2\theta + \sin^2\theta) - 2r\sin\theta\cos\theta \tag{18}$$
$$= r(2 - \sin(2\theta)) \tag{19}$$

$$\left(\frac{\partial z}{\partial \theta}\right)_r = \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial \theta}\right)_r + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial \theta}\right)_r \tag{20}$$
$$= (2x - y)(-r\sin\theta) + (-x + 2y)(r\cos\theta) \tag{21}$$
$$= -r\sin\theta(2r\cos\theta - r\sin\theta) + r\cos\theta(-r\cos\theta + 2r\sin\theta) \tag{22}$$
$$= r^2(\sin^2\theta - 2\sin\theta\cos\theta) + r^2(-\cos^2\theta + 2\sin\theta\cos\theta) \tag{23}$$
$$= r^2(\sin^2\theta - \cos^2\theta) \tag{24}$$
$$= -r^2\cos(2\theta) \tag{25}$$

2. **Solution:** Given a wave u(x, t) which obeys the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{26}$$

one could define two new variables $\zeta = x - ct$, $\eta = x + ct$, so that $x = \frac{1}{2}(\zeta + \eta)$ and $t = \frac{2}{c}(\eta = \zeta)$. By change of variables, $U(\zeta, \eta) = u(x, t)$.

$$\left(\frac{\partial U}{\partial \eta}\right)_\zeta = \left(\frac{\partial u}{\partial x}\right)_t \left(\frac{\partial x}{\partial \eta}\right)_\zeta + \left(\frac{\partial u}{\partial t}\right)_x \left(\frac{\partial t}{\partial \eta}\right)_\zeta \tag{27}$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)_t + \frac{1}{2c}\left(\frac{\partial u}{\partial t}\right)_x. \tag{28}$$

$$\left(\frac{\partial}{\partial \zeta}\left(\frac{\partial U}{\partial \eta}\right)_\zeta\right)_\eta = \frac{1}{2}\left(\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)_t\right)_t \left(\frac{\partial x}{\partial \zeta}\right)_\eta + \left(\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)_t\right)_x \left(\frac{\partial t}{\partial \zeta}\right)_\eta\right) \tag{29}$$

$$+ \frac{1}{2c}\left(\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right)_x\right)_t \left(\frac{\partial x}{\partial \zeta}\right)_\eta + \left(\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)_x\right)_x \left(\frac{\partial t}{\partial \zeta}\right)_\eta\right) \tag{30}$$

$$= \frac{1}{2}\left(\frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \frac{1}{2c}\frac{\partial^2 u}{\partial t \partial x}\right) + \frac{1}{2c}\left(\frac{1}{2}\frac{\partial^2 u}{\partial x \partial t} - \frac{1}{2c}\frac{\partial^2 u}{\partial t^2}\right) \tag{31}$$

$$= \frac{1}{4}\frac{\partial^2 u}{\partial x^2} - \frac{1}{4c^2}\frac{\partial^2 u}{\partial t^2} \tag{32}$$

$$= 0 \tag{33}$$

3. **Solution:** As Pressure P is related to other thermodynamic variables from the equation of state (Van der Waals gas law), so P can be writen as $P(V, T)$, meaning that an exact differential of P exists, given by

$$dP = \left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{\partial P}{\partial V}\right)_T dV \tag{34}$$

From the student's solution, one could identify the purported partial derivatives as $\left(\frac{\partial P}{\partial T}\right)_V = \frac{RT}{V-b}$ and $\left(\frac{\partial P}{\partial V}\right)_T = \frac{RT}{(V-b)^2} - \frac{a}{TV^2}$. As $\frac{\partial^2 P}{\partial T \partial V} = \frac{\partial^2 P}{\partial V \partial T}$, one could check the possibility of the solution by checking whether $\left(\frac{\partial}{\partial V}\left(\frac{RT}{V-b}\right)\right)_T = \left(\frac{\partial}{\partial T}\left(\frac{RT}{(V-b)^2} - \frac{a}{TV^2}\right)\right)_V$.

$$\left(\frac{\partial}{\partial V}\left(\frac{RT}{V-b}\right)\right)_T = -\frac{RT}{(V-b)^2}, \tag{35}$$

$$\left(\frac{\partial}{\partial T}\left(\frac{RT}{(V-b)^2} - \frac{a}{TV^2}\right)\right)_V = \frac{R}{(V-b)^2} + \frac{a}{T^2V^2}. \tag{36}$$

So, the answer from the student is not correct.

4. **Solution:**

(a)

$$C_V(T)dT + \frac{nRT}{V}dV \tag{37}$$

$$\left(\frac{\partial C_V}{\partial V}\right)_T = 0 \tag{38}$$

as $C_V(T)$ is a function of T only.

$$\left(\frac{\partial}{\partial T}\left(\frac{nRT}{V}\right)\right)_V = \frac{nR}{V}. \tag{39}$$

So, the expression is not an exact differential.

(b)

$$\frac{C_V(T)}{T}dT + \frac{nR}{V}dV \tag{40}$$

$$\left(\frac{\partial}{\partial V}\left(\frac{C_V(T)}{T}\right)\right)_T = 0 \tag{41}$$

$$\left(\frac{\partial}{\partial T}\left(\frac{nR}{V}\right)\right)_V = 0 \tag{42}$$

So the expression is an exact differential.

(c)

$$(2xy + y^2)dx + (x^2 + 2xy)dy \tag{43}$$

$$\left(\frac{\partial}{\partial y}(2xy + y^2)\right)_x = 2x + 2y \tag{44}$$

$$\left(\frac{\partial}{\partial x}(x^2 + 2xy)\right)_y = 2x + 2y \tag{45}$$

So the expression is an exact differential, and it can be written as

$$df = (2xy + y^2)dx + (x^2 + 2xy)dy \tag{46}$$

In order to obtain $f(x, y)$, one could perform integration with respect to x and y.

$$f = \int (2xy + y^2)dx \tag{47}$$

$$= x^2y + xy^2 + G(y) \tag{48}$$

$$f = \int (x^2 + 2xy)dy \tag{49}$$

$$= x^2y + xy^2 + H(x) \tag{50}$$

By comparing the two expression, one finds that $G(y) = H(x)$, so they can only equal a constant, so $f = x^2y + xy^2 + C$.

5. **Solution:** Given the van der Waals equation

$$\left(P + \frac{n^2a}{V^2}\right)(V - nb) = nRT \tag{51}$$

To obtain $\left(\frac{\partial V}{\partial T}\right)_{P,n}$, one can perform partial differentiation on both sides.

$$\left(\frac{\partial}{\partial T}\left(\left(P + \frac{n^2a}{V^2}\right)(V - nb)\right)\right)_{P,n} = \left(\frac{\partial}{\partial T}(nRT)\right)_{P,n} \tag{52}$$

$$\left(\frac{\partial}{\partial T}\left(P + \frac{n^2a}{V^2}\right)\right)_{P,n}(V - nb) + \left(P + \frac{n^2a}{V^2}\right)\left(\frac{\partial}{\partial T}(V - nb)\right)_{P,n} = nR \tag{53}$$

$$\frac{-2n^2a(V - nb)}{V^3}\left(\frac{\partial V}{\partial T}\right)_{P,n} + \left(P + \frac{n^2a}{V^2}\right)\left(\frac{\partial V}{\partial T}\right)_{P,n} = nR \tag{54}$$

After some algebraic manipulations, one obtains

$$\left(\frac{\partial V}{\partial T}\right)_{P,n} = nR\left(P - \frac{n^2a}{V^2} + \frac{2n^3ab}{V^3}\right)^{-1}. \tag{55}$$

One can similarly obtain $\left(\frac{\partial V}{\partial P}\right)_{T,n}$.

$$\left(\frac{\partial}{\partial P}\left(\left(P+\frac{n^2a}{V^2}\right)(V-nb)\right)\right)_{T,n} = \left(\frac{\partial}{\partial P}(nRT)\right)_{T,n} \tag{56}$$

$$\left(1-\frac{2n^2a}{V^3}\left(\frac{\partial V}{\partial P}\right)_{T,n}\right)(V-nb)+\left(P+\frac{n^2a}{V^2}\right)\left(\frac{\partial V}{\partial P}\right)_{T,n} = 0 \tag{57}$$

$$(V-nb)-\left(\frac{\partial V}{\partial P}\right)_{T,n}\frac{2n^2a(V-nb)}{V^3}+\left(\frac{\partial V}{\partial P}\right)_{T,n}\left(P+\frac{n^2a}{V^2}\right) = 0 \tag{58}$$

$$\left(\frac{\partial V}{\partial P}\right)_{T,n} = -(V-nb)\left(P-\frac{n^2a}{V^2}+\frac{2n^3ab}{V^3}\right)^{-1} \tag{59}$$

Bt the cyclic rule

$$\left(\frac{\partial P}{\partial T}\right)_{V,n}\left(\frac{\partial T}{\partial V}\right)_{P,n}\left(\frac{\partial V}{\partial P}\right)_{T,n} = -1, \tag{60}$$

$$\left(\frac{\partial P}{\partial T}\right)_{V,n} = -\frac{\left(\frac{\partial V}{\partial T}\right)_{P,n}}{\left(\frac{\partial V}{\partial P}\right)_{T,n}} \tag{61}$$

$$= \frac{nR}{V-nb}. \tag{62}$$

One can also directly compute $\left(\frac{\partial P}{\partial T}\right)_{V,n}$,

$$\left(\frac{\partial}{\partial T}\left(\left(P+\frac{n^2a}{V^2}\right)(V-nb)\right)\right)_{V,n} = \left(\frac{\partial}{\partial T}(nRT)\right)_{V,n} \tag{63}$$

$$\left(\frac{\partial P}{\partial T}\right)_{V,n}(V-nb)+\left(P+\frac{n^2a}{V^2}\right)(0) = nR \tag{64}$$

$$\left(\frac{\partial P}{\partial T}\right)_{V,n} = \frac{nR}{V-nb} \tag{65}$$

As expected, both methods yield the same result.

6. **Solution:** As $u = x^2 + y^2$, du is given by

$$du = 2xdx + 2ydy \tag{66}$$

Under the constraint that u is a constant, $du = 0$. Therefore,

$$0 = 2xdx + 2ydy \tag{67}$$

$$\left(\frac{\partial y}{\partial x}\right)_u = -\frac{x}{y} \tag{68}$$

As shown in the lecture notes,

$$\left(\frac{\partial z}{\partial x}\right)_u = \left(\frac{\partial z}{\partial x}\right)_y + \left(\frac{\partial z}{\partial y}\right)_x\left(\frac{\partial y}{\partial x}\right)_u \tag{69}$$

$$\left(\frac{\partial z}{\partial y}\right)_u = \left(\frac{\partial z}{\partial x}\right)_y\left(\frac{\partial x}{\partial y}\right)_u + \left(\frac{\partial z}{\partial y}\right)_x \tag{70}$$

Substituting in the values of the derivatives $\left(\frac{\partial z}{\partial x}\right)_y = y$, $\left(\frac{\partial z}{\partial y}\right)_x = x$, one obtains

$$\left(\frac{\partial z}{\partial x}\right)_u = y - \frac{x^2}{y} \tag{71}$$

and

$$\left(\frac{\partial z}{\partial y}\right)_u = x - \frac{y^2}{x}. \tag{72}$$